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Universal Arithmetick:

OR, A

TREATISE

OF

ARITHMETICAL

Composition and Resolution.

Univerſal Arithmetick:

OR, A

TREATISE

OF

ARITHMETICAL

Compoſition and Reſolution.

To which is added,

Dr. HALLEY's Method of finding the Roots of Æquations Arithmetically.

Tranſlated from the LATIN *by the late* Mr. RAPHSON, *and reviſed and corrected by* Mr. CUNN.

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TO THE

READER.

TO say any Thing in Praise of the ensuing Treatise, were an Attempt as needless and impertinent, as to write a Panegyrick on its Author. 'Tis enough that the Subject is Algebra; *and that it was written by Sir* Isaac Newton: *Those who know any Thing of the Sciences, need not to be told the Value of the former; nor those who have heard any Thing of Philosophy and Mathematicks, to be instructed in the Praises of the latter. If any Thing could add to the Esteem every Body has for the* Analytick *Art, it must be, that Sir* Isaac *has condescended to handle it; nor could any Thing add to the Opinion the World has of that illustrious*

illustrious Author's Merit, but that he has written with so much Success on that wonderful Subject.

'Tis true, we have already a great many Books of Algebra, *and one might even furnish a moderate Library purely with Authors on that Subject: But as no Body will imagine that Sir* Isaac *would have taken the Pains to compose a new one, had he not found all the old ones defective; so, it will be easily allow'd, that none was more able than he, either to discover the Errors and Defects in other Books, or to supply and rectify them in his own.*

The Book was originally writ for the private Use of the Gentlemen of Cambridge, *and was deliver'd in Lectures, at the publick Schools, by the Author, then* Lucasian *Professor in that University. Thus, not being immediately intended for the Press, the Author had not prosecuted his Subject so far as might otherwise have been expected; nor indeed did he ever find Leisure to bring his Work to a Conclusion: So that it must be observ'd, that all the Constructions, both Geometrical and Mechanical, which occur*

towards

towards the End of the Book, do only serve for finding the first two or three Figures of Roots; the Author having here only given us the Construction of Cubick Æquations, tho' he had a Design to have added, a general Method of constructing Biquadratick, and other higher Powers, and to have particularly shown in what Manner the other Figures of Roots were to be extracted. In this unfinish'd State it continu'd till the Year 1707, *when Mr.* Whiston, *the Author's Successor in the* Lucasian *Chair, considering that it was but small in Bulk, and yet ample in Matter, not too much crowded with Rules and Precepts, and yet well furnish'd with choice Examples, (serving not only as* Praxes *on the Rules, but as Instances of the great Usefulness of the Art itself; and, in short, every Way qualify'd to conduct the young Student from his first setting out on this Study) thought it* Pity *so noble and useful a Work should be doom'd to a College-Confinement, and obtain'd Leave to make it Publick. And in order to supply what the Author had left undone, subjoyn'd the General and truly Noble Method of extracting the Roots of Æquations, publish'd by Dr.*

Dr. Halley *in the* Philosophical Transactions, *having first procur'd both those Gentlemen's Leave for his so doing.*

As to the publishing a Translation of this Book, the Editor is of Opinion, that 'tis enough to excuse his Undertaking, that such Great Men were concern'd in the Original; and is perswaded, that the same Reason which engag'd Sir Isaac *to write, and Mr.* Whiston *to publish the* Latin *Edition, will bear him out in publishing this* English *one: Nor will the Reader require any farther Evidence, that the Translator has done Justice to the Original, after I have assur'd him, that Mr.* Raphson *and Mr.* Cunn *were both concern'd in this Translation.*

Universal

Univerſal Arithmetick;

OR, A

TREATISE

OF

Arithmetical COMPOSITION and RESOLUTION.

COMPUTATION is either perform'd by Numbers, as in Vulgar Arithmetick, or by Species, as uſual among Algebraiſts. They are both built on the ſame Foundations, and aim at the ſame End, *viz.* *Arithmetick* Definitely and Particularly, *Algebra* Indefinitely and Univerſally; ſo that almoſt all Expreſſions that are found out by this Computation, and particularly Concluſions, may be call'd *Theorems*. But *Algebra* is particularly excellent in this, that whereas in *Arithmetick* Queſtions are only reſolv'd by proceeding from given Quantities to the Quantities ſought, *Algebra* proceeds, in a retrograde Order, from the Quantities sought as if they were given, to the Quantities given as if they were sought, to the End that we may some Way or other come to a Conclusion or Æquation, from which one may bring out the Quantity sought. And after this Way the most difficult Problems are resolv'd, the Resolutions whereof would be sought in vain from only common Arithmetick. Yet *Arithmetick* in all its Operations is so subservient to *Algebra*, as that they seem both but to make one perfect Science of Computing; and therefore I will explain them both together.

Whoever goes upon this Science, must first understand the Signification of the Terms and Notes, [or Signs] and learn the fundamental Operations, *viz. Addition, Substraction, Multiplication*, and *Division*; *Extraction of Roots, Reduction of Fractions*, and *Radical Quantities*; and the *Methods of ordering the Terms of Æquations*, and *exterminating the unknown Quantities*, (where they are more than one). Then let [the Learner] proceed to exercise [or put in Practice] these Operations, by bringing Problems to Æquations; and, lastly, let him [learn or] contemplate the Nature and Resolution of Æquations.

Of the Signification of some Words and Notes.

By *Number* we understand not so much a Multitude of Unities, as the abstracted *Ratio* of any Quantity, to another Quantity of the same Kind, which we take for Unity.

[Number] is threefold; integer, fracted, and surd, to which last Unity is incommensurable. Every one understands the Notes of whole Numbers, (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and the Values of those Notes when more than one are set together. But as Numbers plac'd on the left Hand, next before Unity, denote Tens of Units, in the second Place Hundreds, in the third Place Thousands, *&c.* so Numbers set in the first Place after Unity, denote tenth Parts of an Unit, in the second Place hundredth Parts, in the third thousandth Parts, *&c.* and these are call'd *Decimal Fractions*, because they always decrease in a *Decimal Ratio*; and to distinguish the Integers from the Decimals, we place a Comma, or a Point, or a separating Line: Thus the Number 732⌊569 denotes seven hundred thirty two Units, together with five tenth Parts, six centesimal, or hundredth Parts, and nine millesimal, or thousandth Parts of Unity. Which are also written thus 732,⌊569; or thus, 732.569; or also thus, 732⌊569, and so the Number 57104.2083 fifty seven thousand one hundred and four Units,

together

together with two tenth Parts, eight thousandth Parts, and three ten thousandth Parts of Unity; and the Number 0,064 denotes six centesimals and four millesimal Parts. The Notes of Surds and fracted Numbers are set down in the following [Pages].

When the Quantity of any Thing is unknown, or look'd upon as indeterminate, so that we can't express it in Numbers, we denote it by some *Species*, or by some Letter. And if we consider known Quantities as indeterminate, we denote them, for Distinction sake, with the initial [or former] Letters of the Alphabet, as *a*, *b*, *c*, *d*, &c. and the unknown ones by the final ones, *z*, *y*, *x*, &c. Some substitute Consonants or great Letters for known Quantities, and Vowels or little Letters for the unknown ones.

Quantities are either Affirmative, or greater than nothing; or Negative, or less than nothing. Thus in humane Affairs, Possessions or Stock may be call'd *affirmative* Goods, and Debts *negative ones*. And so in local Motion, Progression may be call'd affirmative Motion, and Regression negative Motion; because the first augments, and the other diminishes [the Length of] the Way made. And after the same Manner in Geometry, if a Line drawn any certain Way be reckon'd for Affirmative, then a Line drawn the contrary Way may be taken for Negative: As if AB be drawn to the right, and BC to the left; and AB be reckon'd Affirmative, then BC will be Negative; because in the drawing it diminishes AB, and reduces it either to a shorter, as AC, or to none, if C chances to fall upon the Point A, or to a less than none, if BC be longer than AB from which it is taken [*vide Fig.* 1.] A negative Quantity is denoted by the Sign $-$; the Sign $+$ is prefix'd to an affirmative one; and $\mp$ denotes an uncertain Sign, and $\pm$ a contrary uncertain one.

In an Aggregate of Quantities the Note $+$ signifies, that the Quantity it is prefix'd to, is to be added, and the Note $-$, that it is to be subtracted. And we usually express these Notes by the Words *Plus* (or *more*) and *Minus* (or *less*). Thus $2+3$, or 2 more 3, denotes the Sum of the Numbers 2 and 3, that is 5. And $5-3$, or 5 less 3, denotes the Difference which arises by subducting 3 from 5, that is 2: And $-5+3$ signifies the Difference which arises from subducting 5 from 3, that is 2; and $6-1+3$ makes 8. Also $a+b$ denotes the Sum of the Quantities a and b, and $a-b$ the Difference which arises by subducting b from a; and $a-b+c$ signifies the Sum of that Difference, and of the Quantity c.

 Suppose

Suppose if a be 5, b 2, and c 8, then $a+b$ will be 7, and $a-b$ 3, and $a-b+c$ will be 11. Also $2a+3a$ is $5a$, and $3b-2a-b+3a$ is $2b+a$; for $3b-b$ makes $2b$, and $-1a+3a$ makes $2a$, whose Aggregate, or Sum, is $1b+2a$, and so in others. These Notes $+$ and $-$ are called *Signs*. And when neither is prefix'd, the Sign $+$ is always to be understood.

Multiplication, properly so call'd, is that which is made by Integers, as seeking a new Quantity, so many times greater than the Multiplicand, as the Multiplyer is greater than Unity; but for want of a better Word *Multiplication* is also made Use of in Fractions and Surds, to find a new Quantity in the same *Ratio* (whatever it be) to the Multiplicand, as the Multiplier has to Unity. Nor is Multiplication made only by *abstract* Numbers, but also by *concrete* Quantities, as by Lines, Surfaces, Local Motion, Weights, *&c.* as far as these may be conceiv'd to express [or involve] the same Ratio's to some other known Quantity of the same Kind, esteem'd as Unity, as Numbers do among themselves. As if the Quantity A be to be multiply'd by a Line of 12 Foot, supposing a Line of 2 Foot to be Unity, there will be produc'd by that Multiplication $6A$, or six times A, in the same manner as if A were to be multiply'd by the abstract Number 6; for $6A$ is in the same reason to A, as a Line of 12 Foot has to a Line of 2 Foot. And so if you were to multiply any two Lines, AC and AD, by one another, take AB for Unity, and draw BC, and parallel to it DE, and AE will be the Product of this Multiplication; because it is to AD as AC, to Unity AB, [*vide Fig.* 2.] Moreover, Custom has obtain'd, that the Genesis or Description of a Surface, by a Line moving at right Angles upon another Line, should be called the Multiplication of those two Lines. For tho' a Line, however multiply'd, cannot become a Surface, and consequently this Generation of a Surface by Lines is very different from Multiplication, yet they agree in this, that the Number of Unities in either Line, multiply'd by the Number of Unities in the other, produces an abstracted Number of Unities in the Surface comprehended under those Lines, if the superficial Unity be defin'd as it used to be, *viz.* a Square whose Sides are linear Unities. As if the right Line AB consist of four Unities, and AC of three, then the Rectangle AD will consist of four times three, or 12 square Unities, as from the Scheme will appear, [*vide Fig.* 3.] And there is the like Analogy of a Solid and a Product made by the continual Multiplication of three Quantities. And hence it is, that the Words to *multiply into*, the

Content,

Content, a *Rectangle*, a *Square*, a *Cube*, a *Dimension*, a *Side*, and the like, which are Geometrical Terms, are made Uſe of in Arithmetical Operations. For by a *Square*, or *Rectangle*, or a Quantity of two Dimenſions, we do not always underſtand a Surface, but moſt commonly a Quantity of ſome other Kind, which is produc'd by the Multiplication of two other Quantities, and very often a Line which is produc'd by the Multiplication of two other Lines. And ſo we call a *Cube*, or *Parallelopiped*, or a *Quantity of three Dimenſions*, that which is produc'd by two Multiplications. We ſay likewiſe the *Side* for a *Root*, and uſe *Ducere* in *Latin* inſtead of *Multiply*; and ſo in others.

A Number prefix'd before any Species, denotes that Species to be ſo often to be taken; thus $2a$ denotes two a's, $3b$ three b's, $15x$ fifteen x's. Two or more Species, immediately connected together without any Signs, denote a Product or Quantity made by the Multiplication of all the Letters together. Thus ab denotes a Quantity made by multiplying a by b, and abx denotes a Quantity made by multiplying a by b, and the Product again by x. As ſuppoſe, if a were 2, and b 3, and x 5, then ab would be 6, and abx 30. Among Quantities multiplying one another, take Notice, that the Sign $\times$, or the Word *by* or *into*, is made Uſe of to denote the Product ſometimes; thus 3×5, or 3 by or into 5 denotes 15; but the chief Uſe of theſe Notes is, when compound Quantities are multiply'd together; as if $\overline{y-2b}$ were to multiply $\overline{y+b}$; the Way is to draw a Line over each Quantity, and then write them thus, $\overline{y-2b}$ into $\overline{y+b}$, or $\overline{y-2b}\times\overline{y+b}$.

Diviſion is properly that which is made Uſe of for integer or whole Numbers, in finding a new Quantity ſo much leſs than the Dividend, as Unity is than the Diviſor. But becauſe of the Analogy, the Word may alſo be uſed when a new Quantity is ſought, that ſhall be in any ſuch *Ratio* to the Dividend, as Unity has to the Diviſor, whether that Diviſor be a Fraction or ſurd Number, or other Quantity of any other Kind. Thus to divide the Line AE by the Line AC, AB being Unity, you are to draw ED parallel to CB, and AD will be the Quotient, [*vide Fig.* 4.] Moreover, it is call'd *Diviſion*, by reaſon of the Similitude [it carries with it] when a Rectangle is divided by a given Line as a Baſe, in order thereby to know the Height.

One Quantity below another, with a Line interpos'd, denotes a Quotient, or a Quantity ariſing by the Diviſion of the upper Quantity by the lower. Thus $\frac{6}{2}$ denotes a Quantity arising by dividing 6 by 2, that is 3; and $\frac{5}{8}$ a Quantity arising by the Division of 5 by 8, that is one eighth Part of the Number 5. And $\frac{a}{b}$ denotes a Quantity which arises by dividing a by b; as suppose a was 15 and b 3, then $\frac{a}{b}$ would denote 5. Likewise thus $\frac{ab-bb}{a+x}$ denotes a Quantity arising by dividing $ab-bb$ by $a+x$; and so in others.

These Sorts of Quantities are called *Fractions*, and the upper Part is call'd by the Name of the *Numerator*, and the lower is call'd the *Denominator*.

Sometimes the Divisor is set before the divided Quantity, [or Dividend] and separated from it by [a Mark resembling] an Arch of a Circle. Thus to denote the Quantity which arises by the Division of $\frac{axx}{a+b}$ by $a-b$, we write it thus, $a-b)\frac{axx}{a+b}$.

Altho' we commonly denote Multiplication by the immediate Conjunction of the Quantities, yet an Integer, [set] before a Fraction, denotes the Sum of both; thus $3\frac{1}{2}$ denotes three and a half.

If a Quantity be multiply'd by it self, the Number of *Facts* or Products is, for Shortness sake, set at the Top of the Letter. Thus for aaa we write a^3, for $aaaa$ a^4, for $aaaaa$ a^5, and for $aaabb$ we write a^3bb, or a^3b^2; as, suppose if a were 5 and b be 2, then a^3 will be $5\times5\times5$ or 125, and a^4 will be $5\times5\times5\times5$ or 625, and a^3b^2 will be $5\times5\times5\times2\times2$ or 500. Where Note, that if a Number be written immediately between two Species, it always belongs to the former; thus the Number 3 in the Quantity a^3bb, does not denote that bb is to be taken thrice, but that a is to be thrice multiply'd by it self. Note, moreover, that these Quantities are said to be of so many Dimensions, or of so high a Power or Dignity, as they consist of Factors or Quantities multiplying one another; and the Number set [on forwards] at the top [of the Letter] is called the Index of those Powers or Dimensions; thus aa is [a Quantity] of two Dimensions, or of the 2d Power, and a^3 of three, as the Number 3 at the top denotes. aa is also call'd a *Square*, a^3 a *Cube*, a^4 a [*Biquadrate*, or] *squared Square*, a^5 a *Quadrato-Cube*, a^6 a *Cubo-Cube*, a^7 a *Quadrato-Quadrato-Cube*, [or *Squared-Squared Cube*] and so on. N. B. *Sir* Isaac

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has not here taken any Notice of the more modern Way of expressing these Powers, by calling the Root, or a, *the first* [*or simple*] *Power*, a^2 *the second Power*, a^3 *the third Power*, &c. And the Quantity *a*, by whose Multiplication by it self these Powers are generated, is called their *Root*, *viz.* it is the Square Root of the Square *aa*, the Cube Root of the Cube *aaa*, &c. But when a Root, multiply'd by it self, produces a Square, and that Square, multiply'd again by the Root, produces a Cube, *&c.* it will be (by the Definition of Multiplication) as Unity to the Root; so that Root to the Square, and that Square to the Cube, *&c.* and consequently the Square Root of any Quantity, will be a mean Proportional between Unity and that Quantity, and the Cube Root the first of two mean Proportionals, and the Biquadratick Root the first of three, and so on. Wherefore Roots have these two Properties or Affections, first, that by multiplying themselves they produce the superior Powers; 2dly, that they are mean Proportionals between those Powers and Unity. Thus, 8 is the Square Root of the Number 64, and 4 the Cube Root of it, is hence evident, because 8×8, and $4\times4\times4$ make 64, or because as 1 to 8, so is 8 to 64, and 1 is to 4 as 4 to 16, and as 16 to 64; and hence, if the Square Root of any Line, as *AB*, is to be extracted, produce it to *C*, and let *BC* be Unity; then upon *AC* describe a Semicircle, and at *B* erect a Perpendicular, occurring to [or meeting] the Circle in *D*; then will *BD* be the Root, because it is a mean Proportional between *AB* and Unity *BC*, [*vide Fig.* 5.]

To denote the Root of any Quantity, we use to prefix this Note $\surd$ for a Square Root, and this $\surd 3$ if it be a Cube Root, and this $\surd 4$ for a Biquadratick Root, *&c.* Thus $\surd 64$ denotes 8, and $\surd 3{:}64$ denotes 4; and $\surd aa$ denotes *a*; and $\surd ax$ denotes the Square Root of *ax*; and $\surd 3{:}4axx$ the Cube Root of $4axx$: As if *a* be 3 and *x* 12; then $\surd ax$ will be $\surd 36$, or 6; and $\surd 3{:}4axx$ will be $\surd 3{:}1728$, or 12. And when these Roots can't be [exactly] *found*, or extracted, the Quantities are call'd *Surds*, as $\surd ax$; or *Surd Numbers*, as $\surd 12$.

There are some, that to denote the Square or first Power, make Use of *q*, and of *c* for the Cube, *qq* for the Biquadrate, and *qc* for the Quadrato-Cube, *&c.* After this Manner for the Square, Cube, and Biquadrate of *A*, they write *Aq*, *Ac*, *Aqq*, *&c.* and for the Cube Root of $abb-x^3$, they write $\surd c : abb-x^3$. Others make Use of other Sorts of Notes, but they are now almost out of Fashion.

The

The Mark [or the Sign] $=$ signifies, that the Quantities on each Side of it are equal. Thus $x=b$ denotes x to be equal to b.

The Note $::$ signifies that the Quantities on both Sides of it are Proportional. Thus $a.b :: c.d$ signifies, that a is to b [in the same Proportion] as c to d; and $a.b.e :: c.d.f$ signifies that a, b, and e, are to one another respectively, as c, d, and f, are among themselves; or that a to c, b to d, and e to f, are in the same *Ratio*. Lastly, the Interpretation of any Marks or Signs that may be compounded out of these, will easily be known by the Analogy [they bear to these]. Thus $\frac{3}{4}a^3bb$ denotes three quarters of a^3bb, and $3\frac{a}{c}$ signifies thrice $\frac{a}{c}$, and $7\sqrt{ax}$ seven times $\sqrt{ax}$. Also $\frac{a}{b}x$ denotes the Product of x by $\frac{a}{b}$; and $\frac{5ee}{4a+9e}z^3$ denotes the Product made by multiplying z^3 by $\frac{5ee}{4a+9e}$, that is the Quotient arising by the Division of $5ee$ by $4a+9e$; and $\frac{2a^3}{9c}\sqrt{ax}$, that which is made by multiplying $\sqrt{ax}$ by $\frac{2a^3}{9c}$, and $\frac{7\sqrt{ax}}{c}$ the Quotient arising by the Division of $7\sqrt{ax}$ by c; and $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ the Quotient arising by the Division of $8a\sqrt{cx}$ by the Sum of the Quantities $2a+\sqrt{cx}$. And thus $\frac{3axx-x^3}{a+x}$ denotes the Quotient arising by the Division of the Difference $3axx-x^3$ by the Sum $a+x$, and $\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Root of that Quotient, & $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Product of the Multiplication of that Root by the Sum $2a+3c$. Thus also $\sqrt{\frac{1}{4}aa+bb}$ denotes the Root of the Sum of the Quantities $\frac{1}{4}aa$ and bb, and $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes the Root of the Sum of the Quantities $\frac{1}{2}a$ and $\sqrt{\frac{1}{4}aa+bb}$, and $\frac{2a^3}{aa-zz}\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes that Root multiply'd by $\frac{2a^3}{aa-zz}$, and so in other Cases.

But

But note, that in Complex Quantities of this Nature, there is no Neceſſity of giving a particular Attention to, or bearing in your Mind the Signification of each Letter; it will ſuffice in general to underſtand, *e. g.* that $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ ſignifies the Root of the Aggregate [or Sum] of $\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}$, whatever that Aggregate may chance to be, when Numbers or Lines are ſubſtituted in the Room of Letters. And thus [it is as ſufficient to underſtand] that $\frac{\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}}{a-\sqrt{ab}}$ ſignifies the Quotient ariſing by the Diviſion of the Quantity $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ by the Quantity $a-\sqrt{ab}$, as much as if thoſe Quantities were ſimple and known, though at preſent one may be ignorant what they are, and not give any particular Attention to the Conſtitution or Signification of each of their Parts. Which I thought I ought here [to inſinuate or] admoniſh, leaſt young Beginners ſhould be frighted [or deterr'd] in the very Beginning, by the Complexneſs of the Terms.

Of ADDITION.

THE Addition of Numbers, where they are not compounded, is [eaſy and] manifeſt of it ſelf. Thus it is at firſt Sight evident, that 7 and 9, or 7+9, make 16, and 11+15 make 26. But in [longer or] more compounded Numbers, the Buſineſs is perform'd by writing the Numbers in a Row downwards, or one under another, and ſingly collecting the Sums of the [reſpective] Columns. As if the Numbers 1357 and 172 are to be added, write either of them (ſuppoſe) 172 under the other 1357, ſo that the Units of the one, *viz.* 2, may exactly ſtand under the Units of the other, *viz.* 7, and the other Numbers of the one exactly under the correſpondent ones of the other, *viz.* the Place of Tens under Tens, *viz.* 7 under 5, and that of Hundreds, *viz.* 1, under the Place of Hundreds of the other, *viz.* 3.

$$\begin{array}{r} 1357 \\ 172 \\ \hline 1529 \end{array}$$

Then beginning at the right Hand, ſay, 2 and 7 make 9, which write underneath; alſo 7 and 5 make 12; the laſt of which two Numbers, *viz.* 2, write underneath, and reſerve

 in

in your Mind the other, *viz.* 1, to be added to the two next Numbers, *viz.* 1 and 3; then ſay 1 and 1 make 2, which being added to 3 they make 5, which write underneath, and there will remain only 1, the firſt Figure of the upper Row of Numbers, which alſo muſt be writ underneath; and then you have the whole Sum, *viz.* 1529.

Thus, to add the Numbers 87899 + 13403 + 885 + 1920 into one Sum, write them one under another, ſo that all the Units may make one Column, the Tens another, the Hundredths a third, and the Places of Thouſands a fourth, and ſo on. Then ſay, 5 and 3 make 8, and 8 + 9 make 17; then write 7 underneath, and the 1 add to the next Rank, ſaying 1 and 8 make 9, 9 + 2 make 11, and 11 + 9 makes 20; and having writ the 0 underneath, ſay again as before, 2 and 8 makes 10, and 10 + 9 make 19, and 19 + 4 make 23, and 23 + 8 make 31; then reſerving 3 [in your Memory] write down 1 as before, and ſay again, 3 + 1 make 4, 4 + 3 make 7, and 7 + 7 make 14, wherefore write underneath 4, and laſtly ſay, 1 + 2 make 3, and 3 + 8 make 11, which in the laſt Place write down, and you will have the Sum of them all.

$$\begin{array}{r} 87899 \\ 13403 \\ 1920 \\ 885 \\ \hline 114107 \end{array}$$

After the ſame Manner we alſo add Decimals, as in the following Example may be ſeen:

$$\begin{array}{r} 630,953 \\ 51,0807 \\ 305,27 \\ \hline 987,3037 \end{array}$$

Addition is perform'd in Algebraick Terms, [or Species] by connecting the Quantities to be added with their proper Signs; and moreover, by uniting into one Sum thoſe that can be ſo united. Thus a and b make $a + b$; a and $-b$ make $a - b$; $-a$ and $-b$ make $-a-b$; $7a$ and $9a$ make $7a + 9a$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $-a\sqrt{ac} + b\sqrt{ac}$, or $b\sqrt{ac} - a\sqrt{ac}$; for it is all one, in what Order ſoever they are written.

Affirmative Quantities which agree in [are of the ſame Sort of] Species, are united together, by adding the prefix'd Numbers that are multiply'd into thoſe Species. Thus $7a + 9a$ make $16a$. And $11bc + 15bc$ make $26bc$. Alſo 3

c

+

$+5\frac{a}{c}$ make $8\frac{a}{c}$; and $2\sqrt{ac}+7\sqrt{ac}$ make $9\sqrt{ac}$; and $6\sqrt{ab-xx}+7\sqrt{ab-xx}$ make $13\sqrt{ab-xx}$. And in like manner, $6\sqrt{3}+7\sqrt{3}$ make $13\sqrt{3}$. Moreover, $a\sqrt{ac}+b\sqrt{ac}$ make $\overline{a+b}\sqrt{ac}$, by adding together a and b as Numbers multiplying $\sqrt{ac}$. And so $\frac{\overline{2a+3c}\sqrt{3axx-x^3}}{a+x}+\frac{3a\sqrt{3axx-x^3}}{a+x}$ make $\frac{\overline{5a+3c}\sqrt{3axx-x^3}}{a+x}$ because $2a+3c$ and $3a$ make $5a+3c$.

Affirmative Fractions, that have the same Denominator, are united [or added together] by adding their Numerators. Thus $\frac{1}{5}+\frac{2}{5}$ make $\frac{3}{5}$, and $\frac{2ax}{b}+\frac{3ax}{b}$ make $\frac{5ax}{b}$ and thus $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}+\frac{17a\sqrt{cx}}{2a+\sqrt{cx}}$ make $\frac{25a\sqrt{cx}}{2a+\sqrt{cx}}$, and $\frac{aa}{c}+\frac{bx}{c}$ make $\frac{aa+bx}{c}$.

Negative Quantities are added after the same Way as Affirmative. Thus -2 & -3 make -5; $-\frac{4ax}{b}$ & $-\frac{11ax}{b}$ make $-\frac{15ax}{b}$; $-a\sqrt{ax}$ and $-b\sqrt{ax}$ make $\overline{-a-b}\sqrt{ax}$. But when a Negative Quantity is to be added to an Affirmative one, the Affirmative must be diminish'd by a Negative one. Thus, 3 and -2 make 1; $\frac{11ax}{b}$ and $-\frac{4ax}{b}$ make $\frac{7ax}{b}$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $\overline{b-a}\sqrt{ac}$. And note, that when the Negative Quantity is greater than the Affirmative, the Aggregate [or Sum] will be Negative. Thus 2 and -3 make -1; $-\frac{11ax}{b}$ and $\frac{4ax}{b}$ make $-\frac{7ax}{b}$ and $2\sqrt{ac}$ and $-7\sqrt{ac}$ make $-5\sqrt{ac}$.

In the Addition of a greater Number of Quantities, or more compounded ones, it will be convenient to observe the

the [Method or] Form of Operation we have laid down above in the Addition of Numbers. As if $17ax - 14a + 3$, and $4a + 2 - 8ax$, and $7a - 9ax$, were to be added together, dispose them so in Columns, that the Terms that contain the same Species may stand in a Row one under another, *viz.* the Numbers 3 and 2 in one Column, the Species $-14a$, and $4a$, and $7a$, in another Column, and the Species $17ax$, and $-8ax$, and $-9ax$ in a third; then I add the Terms of each Column by themselves, saying, 2 and 3 make 5, which I write underneath, then $7a$ and $4a$ make $11a$, and moreover $-14a$ make $-3a$, which I also write underneath; lastly, $-9ax$, and $-8ax$ make $-17ax$, to which $17ax$ added makes 0. And so the Sum comes out $-3a + 5$. After the same Manner the Business is done in the following Examples:

$$\begin{array}{rrr} 17ax & -14a & +3 \\ -8ax & +4a & +2 \\ -9ax & +7a & \\ \hline * & -3a & +5 \end{array}$$

$$\begin{array}{rr} 12x & +7a \\ 7x & +9a \\ \hline 19x & +16a \end{array} \qquad \begin{array}{rr} 11bc & -7\sqrt{ac} \\ 15bc & +2\sqrt{ac} \\ \hline 26bc & -5\sqrt{ac} \end{array} \qquad \begin{array}{rrr} -\dfrac{4ax}{b} & +6\sqrt{3} & +\dfrac{1}{5} \\ +\dfrac{11ax}{b} & -7\sqrt{3} & +\dfrac{2}{5} \\ \hline \dfrac{7ax}{b} & -\sqrt{3} & +\dfrac{3}{5} \end{array}$$

$$\begin{array}{rrr} & -6xx & +\frac{3}{7}x \\ 5x^3 & & +\frac{5}{7}x \\ \hline 5x^3 & -6xx & +\frac{8}{7}x \end{array}$$

$$\begin{array}{rrrrr} & & aay & +2a^3 & -\dfrac{a^4}{2y} \\ & -2ayy & -4aay & +a^3 & \\ y^3 & +2ayy & -\frac{1}{2}aay & & \\ \hline y^3 & * & -3\frac{1}{2}aay & +3a^3 & -\dfrac{a^4}{2y} \end{array}$$

$$\begin{array}{rrrr} 5x^4 & +2ax^3 & & \\ -3x^4 & -2ax^3 & +8\frac{1}{4}a^3\sqrt{aa+xx} & \\ -2x^4 & +5bx^3 & & -20a^3\sqrt{aa-xx} \\ & -4bx^3 & -7\frac{1}{4}a^3\sqrt{aa+xx} & \\ \hline * & bx^3 & +a^3\sqrt{aa+xx} & -20a^3\sqrt{aa-xx}. \end{array}$$

Of

Of SUBTRACTION.

THE Invention of the Difference of Numbers [that are] not too much compounded, is of it self evident; as if you take 9 from 17, there will remain 8. But in more compounded Numbers, Subtraction is perform'd by subscribing [or setting underneath] the Subtrahend, and subtracting each of the lower Figures from each of the upper ones. Thus to subtract 63543 from 782579, having subscrib'd 63543, say, 3 from 9 and there remains 6, which write underneath; then 4 from 7 and there remains 3, which write likewise underneath; then 5 from 5 and there remains nothing, which in like manner set underneath; then 3 comes to be taken from 2, but because 3 is greater [than 2] you must borrow 1 from the next Figure 8, which set down, together with 2, makes 12, from which 3 may be taken, and there will remain 9, which write likewise underneath; and then when besides 6 there is also 1 to be taken from 8, add the 1 to the 6, and the Sum 7 [being taken] from 8, there will be left 1, which in like manner write underneath. Lastly, when in the lower [Rank] of Numbers there remains nothing to be taken from 7, write underneath the 7, and so you have the [whole] Difference 719036.

$$\begin{array}{r} 782579 \\ \underline{63543} \\ 719036 \end{array}$$

But especial Care is to be taken, that the Figures of the Subtrahend be [plac'd] or subscrib'd in their [proper or] homogeneous Places; *viz.* the Units of the one under the Units of the other, and the Tens under the Tens, and likewise the Decimals under the Decimals, *&c.* as we have shewn in Addition. Thus, to take the Decimal 0,63 from the Integer 547, they are not to be dispos'd thus $\begin{array}{l} 547 \\ 0{,}63 \end{array}$, but thus $\begin{array}{l} 547 \\ \;\;\;0{,}63 \end{array}$, *viz.* so that the 0's, which supplies the Place of Units in the Decimal, must be plac'd under the Units of the other Number. Then 0 being understood to stand in the empty Places of the upper Number, say, 3 from 0, which since it cannot be, 1 ought to be borrow'd from the foregoing Place, which will make 10, from which 3 is to be taken, and there remains 7, which write underneath. Then that 1 which was borrow'd added to 6 makes 7, and this is to

be

be taken from 0 above it; but since that can't be, you must again borrow 1 from the foregoing Place to make 10; then 7 from 10 leaves 3, which in like manner is to be writ underneath; then that 1 being added to 0, makes 1, which 1 being taken from 7 leaves 6, which again write underneath. Then write the two Figures 54 (since nothing remains to be taken from them) underneath, and you'll have the Remainder 546,37.

$$\begin{array}{r} 547 \\ 0,63 \\ \hline 546,37 \end{array}$$

For Exercise sake, we here set down some more Examples, both in Integers and Decimals:

$$\begin{array}{r} 1673 \\ 1541 \\ \hline 132 \end{array} \qquad \begin{array}{r} 1673 \\ 1580 \\ \hline 93 \end{array} \qquad \begin{array}{r} 458074 \\ 9205 \\ \hline 448869 \end{array} \qquad \begin{array}{r} 35,72 \\ 14,32 \\ \hline 21,4 \end{array} \qquad \begin{array}{r} 46,5003 \\ 3,078 \\ \hline 43,4223 \end{array} \qquad \begin{array}{r} 308,7 \\ 25,74 \\ \hline 282,96 \end{array}$$

If a greater Number is to be taken from a less, you must first subtract the less from the greater, and then prefix a negative Sign to it. As if from 1541 you are to subtract 1673, on the contrary I subtract 1541 from 1673, and to the Remainder 132 I prefix the Sign $-$.

In Algebraick Terms, Subtraction is perform'd by connecting the Quantities, after having chang'd all the Signs of the Subtrahend, and by uniting those together which can be united, as we have done in Addition. Thus $+7a$ from $+9a$ leaves $9a-7a$ or $2a$; $-7a$ from $+9a$ leaves $+9a+7a$, or $16a$; $+7a$ from $-9a$ leaves $-9a-7a$, or $-16a$; and $-7a$ from $-9a$ leaves $-9a+7a$, or $-2a$; so $3\frac{a}{e}$ from $5\frac{a}{e}$ leaves $2\frac{a}{e}$; $7\sqrt{ac}$ from $2\sqrt{ac}$ leaves $-5\sqrt{ac}$; $\frac{2}{9}$ from $\frac{5}{9}$ leaves $\frac{3}{9}$; $-\frac{4}{7}$ from $\frac{3}{7}$ leaves $\frac{7}{7}$; $-\frac{2ax}{b}$ from $\frac{3ax}{b}$ leaves $\frac{5ax}{b}$; $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ from $\frac{-17a\sqrt{cx}}{2a+\sqrt{cx}}$ leaves $\frac{-25a\sqrt{cx}}{2a+\sqrt{cx}}$; $\frac{aa}{c}$ from $\frac{bx}{c}$ leaves $\frac{bx-aa}{c}$; $a-b$ from $2a+b$ leaves $2a+b-a+b$, or $a+2b$; $3az-zz+ac$ from $3az$ leaves $3az-3az+zz-ac$,

or

or $zz - ac$; $\frac{2aa - ab}{c}$ from $\frac{aa + ab}{c}$ leaves $\frac{aa + ab - 2aa + ab}{c}$, or $\frac{-aa + 2ab}{c}$; and $\overline{a - x}\sqrt{ax}$ from $\overline{a + x}\sqrt{ax}$ leaves $\overline{a + x - a + x}\sqrt{ax}$, or $2x\sqrt{ax}$, and so in others. But where Quantities consist of more Terms, the Operation may be manag'd as in Numbers, as in the following Examples:

$$\begin{array}{r} 12x + 7a \\ 7x + 9a \\ \hline 5x - 2a \end{array} \qquad \begin{array}{r} 15bc + 2\sqrt{ac} \\ -11bc + 7\sqrt{ac} \\ \hline 26bc - 5\sqrt{ac} \end{array} \qquad \begin{array}{l} 5x^3 + \frac{5}{7}x \\ 6x^2 - \frac{3}{7}x \\ \hline 5x^3 - 6xx + \frac{5}{7}x \end{array}$$

$$\begin{array}{l} \frac{11ax}{b} - 7\sqrt{3} + \frac{2}{5} \\ \frac{4ax}{b} - 6\sqrt{3} - \frac{1}{5} \\ \hline \frac{7ax}{b} - \sqrt{3} + \frac{3}{5} \end{array}$$

Of MULTIPLICATION.

NUMBERS which arise [or are produc'd] by the Multiplication of any two Numbers, not greater than 9, are to be learnt [and retain'd] in the Memory: As that 5 into 7 makes 35, and that 8 by 9 makes 72, *&c.* and then the Multiplication of greater Numbers is to be perform'd after the same Rule in these Examples.

If 795 is to be multiply'd by 4, write 4 underneath, as you see here. Then say, 4 into 5 makes 20, whose last Figure, *viz.* 0, set under the 4, and reserve the former 2 for the next Operation. Say moreover, 4 into 9 makes 36, to which add the former 2, and there is made 38, whose latter Figure 8 write underneath as before, and reserve the former 3. Lastly, say, 4 into 7 makes 28, to which add the former 3 and there is made 31, which being also set underneath, you'll have the Number 3180, which comes out by multiplying the whole 795 by 4.

$$\begin{array}{r} 795 \\ 4 \\ \hline 3180 \end{array}$$

Moreover,

Moreover, if 9043 be to be multiply'd by 2305, write either of them, *viz.* 2305 under the other 9043 as before, and multiply the *upper* 9043 *first* by 5, after the Manner shewn, and there will come out 45215; then by 0, and there will come out 0000; thirdly, by 3, and there will come out 27129; lastly, by 2, and there will come out 18086. Then dispose these Numbers so coming out in a descending Series, [or under one another] so that the last Figure of every lower Row shall stand one Place nearer to the left Hand than the last of the next superior Row. Then add all these together, and there will arise 20844115, the Number that is made by multiplying the whole 9043 by the whole 2305.

$$\begin{array}{r} 9043 \\ \underline{2305} \\ 45215 \\ 0000 \\ 27129 \\ \underline{18086} \\ 20844115 \end{array}$$

In the same Manner Decimals are multiply'd, by Integers, or other Decimals, or both, as you may see in the following Examples:

$$\begin{array}{r} 72,4 \\ \underline{29} \\ 6516 \\ \underline{1448} \\ 2099,6 \end{array} \qquad \begin{array}{r} 50,18 \\ \underline{2,75} \\ 25090 \\ 35126 \\ \underline{10036} \\ 137,9950 \end{array} \qquad \begin{array}{r} 3,9025 \\ \underline{0,0132} \\ 78050 \\ 117075 \\ \underline{39025} \\ 0,05151300 \end{array}$$

But note, in the Number coming out [or the Product] so many Figures must be cut off to the right Hand for Decimals, as there are Decimal Figures both in the Multiplyer and the Multiplicand. And if by Chance there are not so many Figures in the Product, the deficient Places must be fill'd up to the left Hand with 0's, as here in the third Example.

Simple Algebraick Terms are multiply'd by multiplying the Numbers into the Numbers, and the Species into the Species, and by making the Product Affirmative, if both the Factors are Affirmative, or both Negative; and Negative if otherwise. Thus $2a$ into $3b$, or $-2a$ into $-3b$ make $6ab$, or $6ba$; for it is no Matter in what Order they are plac'd. Thus also $2a$ by $-3b$, or $-2a$ by $3b$ make $-6ab$. And thus, $2ac$ into $8bcc$ make $16abccc$, or $16abc^3$; and $7axx$ into $-12aaxx$ make $-84a^3x^4$; and $-16cy$ into $31ay^3$ make $-496acy^4$; and $-4z$ into

into $-3\sqrt{az}$ make $12z\sqrt{az}$. And ſo 3 into -4 make -12, and -3 into -4 make 12.

Fractions are multiply'd, by multiplying their Numerators by their Numerators, and their Denominators by their Denominators; thus $\frac{2}{5}$ into $\frac{3}{7}$ make $\frac{6}{35}$; and $\frac{a}{b}$ into $\frac{c}{d}$ make $\frac{ac}{bd}$; and $2\frac{a}{b}$ into $3\frac{c}{d}$ make $6 + \frac{a}{b} + \frac{c}{d}$, or $6\frac{ac}{bd}$; and $\frac{3acy}{2bb}$ into $\frac{-7cyy}{4b^3}$ make $\frac{-21accy^3}{8b^5}$; and $\frac{-4z}{c}$ into $\frac{-3\sqrt{az}}{c}$ make $\frac{12z\sqrt{az}}{cc}$; and $\frac{a}{b}x$ into $\frac{c}{d}x^2$ make $\frac{ac}{bd}x^3$. Alſo 3 into $\frac{2}{5}$ make $\frac{6}{5}$, as may appear, if 3 be reduc'd to the Form of a Fraction, *viz.* $\frac{3}{1}$, by making Uſe of Unity for the Denominator. And thus $\frac{15aaz}{cc}$ into $2a$ make $\frac{30a^3z}{cc}$. Whence note by the Way, that $\frac{ab}{c}$ and $\frac{a}{c}b$ are the ſame; as alſo $\frac{abx}{c}$, $\frac{ab}{c}x$, and $\frac{a}{c}bx$, alſo $\frac{\overline{a+b}\sqrt{cx}}{a}$ and $\frac{a+b}{a}\sqrt{cx}$; and ſo in others.

Radical Quantities of the ſame Denomination (that is, if they are both Square Roots, or both Cube Roots, or both Biquadratick Roots, &c.) are multiply'd by multiplying the Terms together [and placing them] under the ſame Radical Sign. Thus $\sqrt{3}$ into $\sqrt{5}$ makes $\sqrt{15}$; and the $\sqrt{ab}$ into $\sqrt{cd}$ makes $\sqrt{abcd}$; and $\sqrt{3:}\,5ayy$ into $\sqrt{3:}\,7ayz$ makes $\sqrt{3:}\,35aay^3z$; and $\sqrt{\frac{a^3}{c}}$ into $\sqrt{\frac{abb}{c}}$ makes $\sqrt{\frac{a^4bb}{cc}}$, that is † $\frac{aab}{c}$; and $2a\sqrt{az}$ into $3b\sqrt{az}$ makes $6ab\sqrt{aazz}$, that is $6aabz$; and $\frac{3xx}{\sqrt{ac}}$ into $\frac{-2x}{\sqrt{ac}}$ makes $\frac{-6x^3}{\sqrt{aacc}}$,

D that

† *See the Chapter of* Notation.

that is $\frac{-6x^3}{ac}$; and $\frac{-4x\sqrt{ab}}{7a}$ into $\frac{-3dd\sqrt{5cx}}{10ee}$ makes $\frac{12ddx\sqrt{5abcx}}{70aee}$.

Quantities that consist of several Parts, are multiply'd by multiplying all the Parts of the one into all the Parts of the other, as is shewn in the Multiplication of Numbers. Thus, $c-x$ into a makes $ac-ax$, and $aa+2ac-bc$ into $a-b$ makes $a^3+2aac-aab-3bac+bbc$. For $aa+2ac-bc$ into $-b$ makes $-aab-2acb+bbc$, and into a makes $a^3+2aac-abc$, the Sum whereof is $a^3+2aac-aab-3abc+bbc$. A Specimen of this Sort of Multiplication, together with other like Examples, you have underneath:

$$\begin{array}{r}
aa+2ac-bc \\
a-b \\
\hline
-aab-2abc+bbc \\
a^3+2aac-abc \\
\hline
a^3+2aac-aab-3abc+bbc
\end{array}
\qquad
\begin{array}{r}
a+b \\
a+b \\
\hline
ab+bb \\
aa+ab \\
\hline
aa+2ab+bb
\end{array}$$

$$\begin{array}{l}
a+b \\
a-b \\
\hline
-ab-bb \\
aa+ab \\
\hline
aa \quad * \quad -bb
\end{array}
\qquad
\begin{array}{r}
yy+2ay-\tfrac{1}{2}aa \\
yy-2ay+aa \\
\hline
aayy+2a^3y-\tfrac{1}{2}a^4 \\
-2ay^3-4aayy+a^3y \\
y^4+2ay^3-\tfrac{1}{2}aayy \\
\hline
y^4 \quad * \quad -3\tfrac{1}{2}aayy+3a^3y-\tfrac{1}{2}a^4
\end{array}$$

$$\begin{array}{c}
\frac{2ax}{c}-\sqrt{\frac{a^3}{c}} \\
3a+\sqrt{\frac{abb}{c}} \\
\hline
\frac{2ax}{c}\sqrt{\frac{abb}{c}}-\frac{aab}{c} \\
\frac{6aax}{c}-3a\sqrt{\frac{a^3}{c}} \\
\hline
\frac{6aax}{c}-3a\sqrt{\frac{a^3}{c}}+\frac{2ax}{c}\sqrt{\frac{abb}{c}}-\frac{aab}{c}.
\end{array}$$

Of

Of Division.

DIVISION is perform'd in Numbers, by seeking how many times the Divisor is contain'd in the Dividend, and as often subtracting, and writing so many Units in the Quotient; and by repeating that Operation upon Occasion, as often as the Divisor can be subtracted. Thus, to divide 63 by 7, seek how many times 7 is contain'd in 63, and there will come out precisely 9 for the Quotient; and consequently $\frac{63}{7}$ *is equal to* 9. Moreover, to divide 371 by 7, prefix the Divisor 7, and beginning at the first Figures of the Dividend, coming as near them as possible, say, how many times 7 is contain'd in 37, and you'll find 5; then writing 5 in the Quotient, subtract 5 × 7, or 35, from 37, and there will remain 2, to which set the last Figure of the Dividend, *viz.* 1; and then 21 will be the remaining Part of the Dividend for the next Operation; say therefore as before, how many times 7 is contain'd in 21? and the Answer will be 3; wherefore writing 3 in the Quotient, take 3 × 7, or 21, from 21 and there will remain 0. Whence it is manifest, that 53 is precisely the Number that arises from the Division of 371 by 7.

```
7) 371 (53
   35
    21
    21
     0
```

And thus to divide 4798 by 23, first beginning with the initial Figures 47, say, how many times is 23 contain'd in 47? Answer 2; wherefore write 2 in the Quotient, and from 47 subtract 2 × 23, or 46, and there will remain 1, to which join the next Number of the Dividend, *viz.* 9, and you'll have 19 to work upon next. Say therefore, how many times is 23 contain'd in 19? Answer 0; wherefore write 0 in the Quotient; and from 19 subtract 0 × 23, or 0, and there remains 19, to which join the last Number 8, and you'll have 198 to work upon next. Wherefore

```
23) 4798 (208,6086, &c.
    46
     19
     00
     198
     184
      140
      138
        20
        00
        200
        184
         160
```

in the laſt Place ſay, how many times is 23 contain'd in 198 (which may be gueſs'd at from the firſt Figures of each, 2 and 19, by taking notice how many times 2 is contain'd in 19)? I anſwer 8; wherefore write 8 in the Quotient, and from 198 ſubtract 8×23, or 184, and there will remain 14 to be farther divided by 23; and ſo the Quotient will be $208\frac{14}{23}$. And if this Fraction is not lik'd, you may continue the Diviſion in Decimal Fractions as far as you pleaſe, by adding always a Cypher to the remaining Number. Thus to the Remainder 14 add 0, and it becomes 140. Then ſay, how many times 23 in 140? Anſwer 6; write therefore 6 in the Quotient; and from 140 ſubtract 6×23, or 138, and there will remain 2; to which ſet a Cypher (or 0) as before. And thus the Work being continu'd as far as you pleaſe, there will at length come out this Quotient, *viz.* 208,6086, &c.

After the ſame Manner the Decimal Fraction 3,5218 is divided, by the Decimal Fraction 46,1, and there comes out 0,07639, *&c.* Where note, that there *muſt* be ſo *many Figures* cut off in the Quotient, for Decimals, as there are more in the laſt Dividend than the Diviſor: As in this Example 5, becauſe there are 6 in the laſt Dividend, *viz.* 3,521800, and 1 in the Diviſor 46,1.

```
46,1) 3,5218 (0,07639
      322,7
      -----
       2948
       2766
       ----
        1820
        1383
        ----
         4370
         4149
         ----
          221
```

We have here ſubjoin'd more Examples, for Clearneſs ſake, *viz.*

```
9043) 20844115 (2305.          73,4) 2099,6 (29
      18086                            1448
      -----                            ----
       27581                           6516
       27129                           6516
       -----                           ----
        45215                             0
        45215
        -----
            0
```

```
50,18) 137,995 (2,75
       10036
       -----
        37635
        35126
        -----
         25090
         25090
         -----
             0
```

```
0,0132) 0,051513 (3,9025
          396
          ----
          1191
          1188
          ----
           330
           264
           ---
            660
            660
            ---
              0
```

N. B. *The wording of this Rule in Sir* Isaac, *seeming a little obscure, this other equivalent Rule may be added*, viz. *Observe what is the Quality of that Figure in the Dividend under which the Place of Integer Units in the Divisor does or should stand; for the same will be the Quality of the first Figure of the Quotient*, e. g.

345) ,00468 (1

In this Example, 5 being the Place of Integer Units in the Dividend, that set under the Dividend, so as to divide it, would fall under the Figure 8, which is the Place of Hundreds of Thousandths in the Dividend; therefore the Unit in the Quotient must stand in the Place of Hundreds of Thousandths; and to make it do so, four Cyphers must be plac'd before it, viz. ,00001, &c. *is the true Quotient.*

In Algebraick Terms Division is perform'd by the Resolution of what is compounded by Multiplication. Thus, ab divided by a gives for the Quotient b. $6ab$ divided by $2a$ gives $3b$; and divided by $-2a$ gives $-3b$. $-6ab$ divided by $2a$ gives $-3b$, and divided by $-2a$ gives $3b$. $16abc^3$ divided by $2ac$ gives $8bcc$. $-84a^3x^4$ divided by $-12aaxx$ gives $7axx$. Likewise $\frac{6}{35}$ divided by $\frac{2}{5}$ gives $\frac{3}{7}$. $\frac{ac}{bd}$ divided by $\frac{a}{b}$ gives $\frac{c}{d}$. $\frac{-21accy^3}{8b^5}$ divided by $\frac{3acy}{2bb}$ gives $\frac{-7cyy}{4b^3}$. $\frac{6}{5}$ divided by 3 gives $\frac{2}{5}$; and reciprocally $\frac{6}{5}$ divided by $\frac{2}{5}$ gives $\frac{3}{1}$, or 3. $\frac{30a^3z}{cc}$ divided

divided by $2a$ gives $\frac{15aaz}{cc}$; and reciprocally divided by $\frac{15aaz}{cc}$ gives $2a$. Likewise $\sqrt{15}$ divided by $\sqrt{3}$ gives $\sqrt{5}$. $\sqrt{abcd}$ divided by $\sqrt{cd}$ gives $\sqrt{ab}$. $\sqrt{a^3c}$ by $\sqrt{ac}$ gives $\sqrt{aa}$, or a. $\sqrt{3:\,35aay^3z}$ divided by $\sqrt{3:\,5ayy}$ gives $\sqrt{3:\,7ayz}$. $\frac{\sqrt{a^4bb}}{cc}$ divided by $\frac{\sqrt{a^3}}{c}$ gives $\frac{\sqrt{abb}}{c}$. $\frac{12ddx\sqrt{5abcx}}{70aee}$ divided by $\frac{-3dd\sqrt{5cx}}{10ee}$ gives $\frac{-4x\sqrt{ab}}{7a}$. And so $\overline{a+b}\sqrt{ax}$ divided by $a+b$ gives $\sqrt{ax}$; and reciprocally divided by $\sqrt{ax}$ gives $a+b$. And $\frac{a}{a+b}\sqrt{ax}$ divided by $\frac{1}{a+b}$ gives $a\sqrt{ax}$, or divided by a gives $\frac{1}{a+b}\sqrt{ax}$, or $\frac{\sqrt{ax}}{a+b}$; and reciprocally divided by $\frac{\sqrt{ax}}{a+b}$ gives a. But in Divisions of this Kind you are to take care, that the Quantities divided by one another be of the same Kind, *viz.* that Numbers be divided by Numbers, and Species by Species, Radical Quantities by Radical Quantities, Numerators of Fractions by Numerators, and Denominators by Denominators; also in Numerators, Denominators, and Radical Quantities, the Quantities of each Kind must be divided by homogeneous ones [or Quantities of the same Kind.]

Now if the Quantity to be divided cannot be divided by the Divisor [propos'd], it is sufficient to write the Divisor underneath, with a Line between them. Thus to divide ab by c, write $\frac{ab}{c}$; and to divide $\overline{a+b}\sqrt{cx}$ by a, write $\frac{\overline{a+b}\sqrt{cx}}{a}$, or $\frac{a+b}{a}\sqrt{cx}$. And so $\sqrt{ax-xx}$ divided by $\sqrt{cx}$ gives $\frac{\sqrt{ax-xx}}{\sqrt{cx}}$, or $\sqrt{\frac{ax-cx}{cx}}$. And $\overline{aa+ab}\sqrt{aa-2xx}$ divided by $\overline{a-b}\sqrt{aa-xx}$ gives $\frac{aa+ab}{a-b}\sqrt{\frac{ax-2xx}{aa-xx}}$. And $12\sqrt{5}$ divided by $4\sqrt{7}$ gives $3\sqrt{\frac{5}{7}}$.

But

But when these Quantities are Fractions, multiply the Numerator of the Dividend into the Denominator of the Divisor, and the Denominator into the Numerator, and the first Product will be the Numerator, and the latter the Denominator of the Quotient. Thus to divide $\frac{a}{b}$ by $\frac{c}{d}$ write $\frac{ad}{bc}$, that is, multiply a by d and b by c. In like Manner, $\frac{3}{7}$ by $\frac{5}{4}$ gives $\frac{12}{35}$. And $\frac{3a}{4c}\sqrt{ax}$ divided by $\frac{2c}{5a}$ gives $\frac{15aa}{8cc}\sqrt{ax}$, and divided by $2c\frac{\sqrt{aa-xx}}{5a\sqrt{ax}}$ gives $\frac{15a^3x}{8cc\sqrt{aa-xx}}$. After the same Manner, $\frac{ad}{b}$ divided by c (or by $\frac{c}{1}$) gives $\frac{ad}{bc}$. And c (or $\frac{c}{1}$) divided by $\frac{ad}{b}$ gives $\frac{bc}{ad}$. And $\frac{3}{7}$ divided by 5 gives $\frac{3}{35}$. And 3 divided by $\frac{5}{4}$ gives $\frac{12}{5}$. And $\frac{a+b}{c}\sqrt{cx}$ divided by a gives $\frac{a+b}{ac}\sqrt{cx}$. And $\overline{a+b}\sqrt{cx}$ divided by $\frac{a}{c}$ gives $\frac{ac+bc}{a}\sqrt{cx}$. And $2\sqrt{\frac{axx}{c}}$ divided by $3\sqrt{cd}$ gives $\frac{2}{3}\sqrt{\frac{axx}{ccd}}$; and divided by $3\sqrt{\frac{cd}{x}}$ gives $\frac{2}{3}\sqrt{\frac{ax^3}{ccd}}$. And $\frac{1}{5}\sqrt{\frac{7}{11}}$ divided by $\frac{1}{2}\sqrt{\frac{3}{7}}$ gives $\frac{2}{5}\sqrt{\frac{49}{33}}$, and so in others.

A Quantity compounded of several Terms, is divided by dividing each of its Terms by the Divisor. Thus $aa+3ax-xx$ divided by a gives $a+3x-\frac{xx}{a}$. But when the Divisor consists also of several Terms, the Division is perform'd as in Numbers. Thus to divide $a^3+2aac-aab-3abc+bbc$ by $a-b$, say, how many times is a contain'd in a^3, *viz.* the first Term of the Divisor in the first Term of the Dividend? Answer aa. Wherefore write aa in the Quotient; and having subtracted $a-b$ multiply'd into aa, or a^3-aab from the Dividend, there will remain

$2aac$

$2aac - 3abc + bbc$ yet to be divided. Then ſay again, how many times a in $2aac$? Anſwer $2ac$. Wherefore write alſo $2ac$ in the Quotient, and having ſubtracted $a - b$ into $2ac$, or $2aac - 2abc$ from the aforeſaid Remainder, there will yet remain $-abc + bbc$. Wherefore ſay again, how many times a in $-abc$? Anſwer $-bc$, and then write $-bc$ in the Quotient; and having, in the laſt Place, ſubtracted $+a-b$ into $-bc$, *viz.* $-abc + bbc$ from the laſt Remainder, there will remain nothing; which ſhews that the Diviſion is at an End, and the Quotient coming out [juſt] $aa + 2ac - bc$.

But that theſe Operations may be duly reduc'd to the Form which we uſe in the Diviſion of Numbers, the Terms both of the Dividend and the Diviſor muſt be diſpos'd in Order, according to the Dimenſions of that Letter which is [ofteneſt found or] judg'd moſt proper for the [Eaſe of the] Operation; ſo that thoſe Terms may ſtand firſt, in which that Letter is of moſt Dimenſions, and thoſe in the ſecond Place whoſe Dimenſions are next higheſt; and ſo on to thoſe wherein that Letter is not at all involv'd, [or into which it is not at all multiply'd] which ought to ſtand in the laſt Place. Thus, in the Example we juſt now brought, if the Terms are diſpos'd according to the Dimenſions of the Letter a, the following Diagram will ſhew the Form of the Work, *viz.*

$$
\begin{array}{llllll}
a-b) & a^3 & \begin{matrix}+2aac\\-aab\end{matrix} & -3abc & +bbc & (aa+2ac-bc \\
\hline
 & a^3 & -aab & & & \\
\hline
 & 0 & +2aac & -3abc & & \\
 & & 2aac & -2abc & & \\
\hline
 & & 0 & -abc & +bbc & \\
 & & & -abc & +bbc & \\
\hline
 & & & 0 & 0 &
\end{array}
$$

Where may be ſeen, that the Term a^3, or a of three Dimenſions, ſtands in the firſt Place of the Dividend, and the Terms $\frac{2aac}{-aab}$, in which a is of two Dimenſions, ſtand in the ſecond Place, and ſo on. The Dividend might alſo have been writ thus;

$$a^3 \begin{matrix}+2c\\-b\end{matrix} aa - 3bca + bbc.$$

Where the Terms that ſtand in the ſecond Place are united, by collecting together [or placing by one another] the Factors [or Coefficients] of the Letter [where it is] of the ſame Dimenſion. And thus, if the Terms were to be diſpos'd according to the Dimenſions of the Letter b, the Buſineſs muſt be perform'd [or would ſtand] as in the following Diagram, the Explication whereof we ſhall here ſubjoin:

$$\begin{array}{llll}
-b+a)\; cbb & \begin{matrix}-3ac\\-aa\end{matrix}\, b & \begin{matrix}+a^3\\+2aac\end{matrix} & \left(-cb \begin{matrix}+2ac\\+aa\end{matrix}\right. \\
\hline
\quad cbb & -acb & & \\
\hline
\qquad 0 & \begin{matrix}-2ac\\-aa\end{matrix}\, b & \begin{matrix}+a^3\\+2aac\end{matrix} & \\
 & \begin{matrix}-2ac\\-aa\end{matrix}\, b & \begin{matrix}+2aac\\+a^3\end{matrix} & \\
\hline
 & \quad 0 & \quad 0 &
\end{array}$$

Say, How many times is $-b$ contain'd in cbb? Anſwer $-cb$. Wherefore having writ $-cb$ in the Quotient, ſubtract $\overline{-b+a} \times -cb$, or $bbc - abc$, and there will remain in the ſecond Place $\begin{matrix}-2ac\\-aa\end{matrix}\, b$. To this Remainder add, if you pleaſe, the Quantities [that ſtand] in the laſt Place, viz. $\begin{matrix}a^3\\2aac\end{matrix}$, and ſay again, how many times is $-b$ contain'd in $\begin{matrix}-2ac\\-aa\end{matrix}\, b$? Anſwer $\begin{matrix}+2ac\\+aa\end{matrix}$. Theſe therefore being writ in the Quotient, ſubtract $-b+a$ multiply'd by $\begin{matrix}+2ac\\+aa\end{matrix}$, or $\begin{matrix}-2ac\\-aa\end{matrix}\, b \begin{matrix}+2aac\\+a^3\end{matrix}$, and there will remain nothing. Whence it is manifeſt, that the Diviſion is at an End, the Quotient coming out $-cb + 2ac + aa$, as before.

And thus, if you were to divide $aay^4 - aac^4 + yyc^4 + y^6 - 2y^4cc - a^6 - 2a^4cc - a^4yy$ by $yy - aa - cc$. I order [or place] the Quantities according to the [Dimenſions of the] Letter y, thus:

$$yy \begin{matrix} -aa \\ -cc \end{matrix} \Big) \, y^6 \begin{matrix} +aa \\ -2cc \end{matrix} y^4 \begin{matrix} -a^4 \\ +c^4 \end{matrix} yy \begin{matrix} -a^6 \\ -2a^4cc. \\ -aac^4 \end{matrix}$$

Then I divide as in the following Diagram.

Here are added other Examples, in which you are to take Notice, that where the Dimensions of the Letter, which [this Method of] ordering ranges, don't always proceed in the same Arithmetical Progression, but sometimes [interruptedly, or] by Way of Skipping, in the defective Places we note [this Mark] *.

$$yy \begin{matrix} -aa \\ -cc \end{matrix} \Big) \, y^6 \begin{matrix} +aa \\ -2cc \end{matrix} y^4 \begin{matrix} -a^4 \\ +c^4 \end{matrix} yy \begin{matrix} -a^6 \\ -2a^4cc. \\ -aac^4 \end{matrix}$$

$$y^6 \begin{matrix} -aa \\ -cc \end{matrix} y^4 \qquad \Big(\, y^4 \begin{matrix} +2aa \\ -cc \end{matrix} y^2 \begin{matrix} +a^4 \\ +aacc. \end{matrix}$$

$$0 \begin{matrix} +2aa \\ -cc \end{matrix} y^4 \begin{matrix} -2a^4 \\ -aacc \\ +c^4 \end{matrix} y^2$$

$$0 \qquad \begin{matrix} +a^4 \\ +aacc \end{matrix} y^2$$

$$\begin{matrix} +a^4 \\ +aacc \end{matrix} y^2 \begin{matrix} -a^6 \\ -2a^4cc \\ -aac^4 \end{matrix}$$

$$0 \qquad 0$$

$$a+b) \; aa * -bb \; (a-b$$

$$aa+ab$$

$$0-ab$$

$$-ab-bb$$

$$0 \qquad 0$$

$$\begin{array}{l}
yy - 2ay + aa)\quad y^4 * - 3\tfrac{1}{2}aayy + 3a^3y - \tfrac{1}{2}a^4 \quad (yy + 2ay - \tfrac{1}{2}aa \\
\underline{y^4 - 2ay^3 + aayy} \\
0 + 2ay^3 - 4\tfrac{1}{2}aayy \\
\underline{+ 2ay^3 - 4\,aayy + 2a^3y} \\
\quad 0 - \tfrac{1}{2}aayy + a^3y \\
\quad \underline{- \tfrac{1}{2}aayy + a^3y - \tfrac{1}{2}a^4} \\
\qquad\quad 0 \quad 0 \quad 0
\end{array}$$

$$\begin{array}{l}
aa + ab\sqrt{2} + bb)\quad a^4 \quad * \quad * \quad * \quad + b^4 \quad (aa - ab\sqrt{2} + bb \\
\underline{a^4 + a^3b\sqrt{2} + aabb} \\
- a^3b\sqrt{2} - aabb \\
\underline{- a^3b\sqrt{2} - 2aabb - ab^3\sqrt{2}} \\
\qquad + aabb + ab^3\sqrt{2} \\
\qquad \underline{+ aabb + ab^3\sqrt{2} + b^4} \\
\qquad\quad 0 \quad 0 \quad 0
\end{array}$$

Some begin Division from the last Terms, but it comes to the same Thing, if, inverting the Order of the Terms, you begin from the first. There are also other Methods of dividing, but it is sufficient to know the most easy and commodious.

Of EXTRACTION *of* ROOTS.

WHEN the Square Root of any Number is to be extracted, it is first to be noted with Points in every other Place, beginning from Unity; then you are to write down such a Figure for the Quotient, or Root, whose Square shall be equal to, or nearest, less than the Figure or Figures to the first Point. And [then] subtracting that Square, the other Figures of the Root will be found one by one, by dividing the Remainder by the double of the Root as far as extracted, and each Time taking from that Remainder the

Square of the Figure that laſt came out, and the Decuple of the aforeſaid Diviſor augmented by that Figure.

Thus to extract the Root out of 99856, firſt Point it after this Manner, 99856, then ſeek a Number whoſe Square ſhall equal the firſt Figure 9, *viz.* 3, and write it in the Quotient; and then having ſubtracted from 9, 3×3, or 9, there will remain 0; to which ſet down the Figures to the next Point, *viz.* 98 for the following Operation. Then taking no Notice of the laſt Figure 8, ſay, How many times is the Double of 3, or 6, contain'd in the firſt Figure 9? Anſwer 1; wherefore having writ 1 in the Quotient, ſubtract the Product of 1 × 61, or 61, from 98, and there will remain 37, to which connect the laſt Figures 56, and you'll have the Number 3756, in which the Work is next to be carry'd on. Wherefore alſo neglecting the laſt Figure of this, *viz.* 6, ſay, How many times is the double of 31, or 62, contain'd in 375, (which is to be gueſs'd at from the initial Figures 6 and 37, by taking Notice how many times 6 is contain'd in 37?) Anſwer 6; and writing 6 in the Quotient, ſubtract 6×626, or 3756, and there will remain 0; whence it appears that the Buſineſs is done; the Root coming out 316.

```
99856 (316
9
-----
098
 61
-----
 3756
 3756
-----
    0
```

Otherwiſe with the Diviſors ſet down it will ſtand thus:

```
    99856 (316
    9
    -----
  6)98
    61
    -----
 62)3756
    3756
    -----
       0
```

And ſo in others.

And ſo if you were to extract the Root out of 22178791, firſt having pointed it, ſeek a Number whoſe Square (if it cannot be [exactly] equall'd) ſhall be the next leſs Square (or neareſt) to 22, the Figures to the firſt Point, and you'll find

find it to be 4. For 5×5, or 25, is greater than 22; and 4×4, or 16, less; wherefore 4 will be the first Figure of the Root. This therefore being writ in the Quotient, from 22 take the Square 4×4, or 16, and to the Remainder 6 adjoin moreover the next Figures 17, and you'll have 617, from whose Division by the double of 4 you are to obtain the second Figure of the Root, *viz.* neglecting the last Figure 7, say, how many times is 8 contain'd in 61? Answer 7; wherefore write 7 in the Quotient, and from 617 take the Product of 7 into 87, or 609, and there will remain 8, to which join the two next Figures 87, and you'll have 887, by the Division whereof by the double of 47, or 94, you are to obtain the third Figure; as say, How many times is 94 contain'd in 88? Answer 0; wherefore write 0 in the Quotient, and adjoin the two last Figures 91, and you'll have 88791, by whose Division by the double of 470, or 940, you are to obtain the last Figure, *viz.* say, How many times 940 in 8879? Answer 9; wherefore write 9 in the Quotient, and you'll have the Root 4709.

```
22178791 (4709,43637, &c.
16
--
 617
 609
 ---
   88791
   84681
   -----
    411000
    376736
    ------
     3426400
     2825649
     -------
      60075100
      56513196
      --------
       356190400
       282566169
       ---------
        73624231
```

But since the Product 9×9409, or 84681, subtracted from 88791, leaves 4110, that is a Sign that the Number 4709 is not the Root of the Number 22178791 precisely, but that it is a little less. And in this Case, and in others like it, if you desire the Root should approach nearer, you must [proceed or] carry on the Operation in Decimals, by adding to the Remainder two Cyphers in each Operation. Thus the Remainder 4110 having two Cyphers added to it, becomes 411000; by the Division whereof by the double of 4709, or 9418; you'll have the first Decimal Figure 4. Then having writ 4 in the Quotient, subtract 4×94184, or 376736 from 411000, and there will remain 34264. And so so having added two more Cyphers, the Work may be carry'd on at Pleasure, the Root at length coming out 4709,43637, &c.

But when the Root is carry'd on half-way, or above, the rest of the Figures may be obtain'd by Division alone. As in this Example, if you had a Mind to extract the Root to nine Figures, after the five former 4709,4 are extracted, the four latter may be had, by dividing the Remainder by the double of 4709,4.

And after this Manner, if the Root of 32976 was to be extracted to five Places in Numbers: After the Figures are pointed, write 1 in the Quotient, as [being the Figure] whose Square 1×1, or 1, is the greatest that is contain'd in 3 the Figure to the first Point; and having taken the Square of 1 from 3, there will remain 2; then having set the two next Figures, *viz.* 29 to it, (*viz.* to 2) seek how many times the double of 1, or 2, is contain'd in 22, and you'll find indeed that it is contain'd more than 10 times; but you are *never to take your Divisor 10 times*, no, nor 9 times in this Case; because the Product of 9×29, or 261, is greater than 229, from which it would be to be taken [or subtracted]. Wherefore write only 8. And *then having writ 8 in the Quotient*, and subtracted 8×28, or 224, there will remain 5; and having set down to this the Figures 76, seek how many times the double of 18, or 36, is contain'd in 57, and you'll find 1, and so write 1 in the Quotient; and having subtracted 1×361, or 361 from 576, there will remain 215. Lastly, to obtain the remaining Figures, divide this Number 215 by the double of 181, or 362, and you'll have the Figures 59, which being writ *in the Quotient, you'll have the Root* 181,59.

```
    32976 (181,59
    1
    ---
2)  229
    224
    ---
36) 576
    361
    ---
362)215 (59, &c.
```

After the same Way Roots are also extracted out of Decimal Numbers. Thus the Root of 329,76 is 18,159; and the Root of 3,2976 is 1,8159; and the Root of 0,032976 is 0,18159, and so on. But the Root of 3297,6 is 57,4247; and the Root of 32,976 is 5,74247. And thus the Root of 9,9856 is 3,16. But the Root of 0,99856 is 0,999279, &c. as will appear from the following Diagrams:

```
        3297,60 (57,4247
        25
  10)   797
        749
  114)  4860
        4576
  1148) 28400
        22964
  11484)  543600
          459376
  114848) 8422400
          8039409
           382991
```

```
     9,9856 (3,16
     9
  6) 98
     61
 62) 3756
     3756
        0
```

```
         0,998560 (0,999279
          81
  18)     1885
          1701
  198)    18460
          17901
  1998)   55900
          39964
  19984)  1593600
          1398929
  199854) 19467100
          17986941
           1480159
```

I will comprehend the Extraction of the Cubick Root, and of all others, under one general Rule, consulting rather the Ease of the Praxis than the Expeditiousness of it, lest I should [too much] retard [the Learner] in Things that are of no frequent Use, *viz.* every third Figure beginning from Unity is first of all to be pointed, if the Root [to be extracted] be a Cubick one; or every fifth, if it be a Quadrato-Cubick [or of the fifth Power], and then such a Figure is to be writ in the Quotient, whose greatest Power (*i. e.* whose Cube, if it be a Cubick Power, or whose Quadrato-Cube, if it be the fifth Power, *&c.*) shall either be equal to to the Figure or Figures before the first Point, or next less [under them]; and then having subtracted that Power, the next Figure will be found by dividing the Remainder augmented by the next Figure of the Resolvend, by the next least Power of the Quotient, multiply'd by the Index of the Power to be extracted, that is, by the triple Square, if the Root be a Cubick one; or by the quintuple Biquadrate [*i. e.* five times the Biquadrate] if the Root be of the fifth Power, *&c.* And having again subtracted the Power of the whole Quotient from the first Resolvend, the third Figure will be found by dividing that Remainder augmented by the next Figure of the Resolvend, by the next least Power of the whole Quotient, multiply'd by the Index of the Power to be extracted.

Thus to extract the Cube Root of 13312053, the Number is first to be pointed after this Manner, *viz.* $1\dot{3}31\dot{2}05\dot{3}$. Then you are to write the Figure 2, whose Cube is 8, in the [first Place of] the Quotient, as which is the next least [Cube] to the Figures 13, [which is not a perfect Cube Number] or to the first Point; and having subtracted that Cube, there will remain 5; which being augmented by the next Figure of the Resolvend 3, and divided by the triple Square of the Quotient 2, by seeking how many times 3×4, or 12, is contain'd in 53, it gives 4 for the second Figure of the Quotient. But since the Cube of the Quotient 24, *viz.* 13824 would come out too great to be subtracted from the Figures 13312 that precede the second Point, there must only 3 be writ in the Quotient: Then the Quotient 23 being in a separate Paper, [or Place] multiply'd by 23 gives the Square 529, which again multiply'd by 23 gives the Cube 12167, and this taken from 13312, will leave 1145; which augmented by the next Figure of the Resolvend 0, and divided by the triple Square of the Quotient 23, *viz.* by seeking how many times 3×529, or 1587, is contain'd in 11450, it gives 7 for the third Figure of the Quotient. Then the Quotient 237, multiply'd by 237, gives the

```
                     13312053 (237
Subtract the Cube    8
               12) rem. 53 (4 or 3
      Subtract Cube 12167
         1587) rem. 11450 (7
          Subtract 13312053
                   Remains 0
```

the Square 56169, which again multiply'd by 237 gives the Cube 13312053, and this taken from the Reſolvend leaves 0. Whence it is evident that the Root ſought is 237.

And ſo to extract the Quadrato-Cubical Root of 36430820, it muſt be pointed over every fifth Figure, and the Figure 3, whoſe Quadrato-Cube [or fifth Power] 243 is the next leaſt to 364, *viz.* to the firſt Point, muſt be writ in the Quotient. Then the Quadrato-Cube 243 being ſubtracted from 364, there remains 121, which augmented by the next Figure of the Reſolvend, *viz.* 3, and divided by five times the Biquadrate of the Quotient, *viz.* by ſeeking how many times 5×81, or 405, is contain'd in 1213, it gives 2 for the ſecond Figure. That Quotient 32 being thrice multiply'd by it ſelf, makes the Biquadrate 1048576; and this again multiply'd by 32, makes the Quadrato-Cube 33554432, which being ſubtracted from the Reſolvend leaves 2876388. Therefore 32 is the Integer Part of the Root, but not the true Root; wherefore, if you have a Mind to proſecute the Work in Decimals, the Remainder, augmented by a Cypher, muſt be divided by five times the aforeſaid Biquadrate of the Quotient, by ſeeking how many times 5×1048576, or 5242880, is contain'd in 2876388,0, and there will come out the third Figure, or the firſt Decimal 5. And ſo by ſubtracting the Quadrato-Cube of the Quotient 32,5 from the Reſolvend, and dividing the Remainder by five times its Biquadrate, the fourth Figure may be obtain'd. And ſo on *in Infinitum.*

$$\begin{array}{rl} & 36430820\ (32,5 \\ & 243 \\ 405) & 1213\ (2 \\ & 33554432 \\ 5242880) & 2876388,0\ (5 \end{array}$$

When the Biquadratick Root is to be extracted, you may extract twice the Square Root, becauſe $\surd^4$ is as much as $\surd^2 \surd^2$. And when the Cubo-Cubick Root is to be extracted, you may firſt extract the Cube-Root, and then the Square-Root of that Cube-Root, becauſe the $\surd^6$ is the ſame as $\surd^3 \surd^3$; whence ſome have call'd theſe Roots not Cubo-Cubick ones, but Quadrato-Cubes. And the ſame is to be obſerv'd in other Roots, whoſe Indexes are not prime Numbers.

The Extraction of Roots out of ſimple Algebraick Quantities, is evident, even from [the Nature or Marks of] Notation it ſelf; as that $\surd aa$ is a, and that $\surd aacc$ is ac, and

that

that $\sqrt{9aacc}$ is $3ac$, and that $\sqrt{49a^4xx}$ is $7aax$. And also that $\sqrt{\frac{a^4}{cc}}$, or $\frac{\sqrt{a^4}}{\sqrt{cc}}$ is $\frac{aa}{c}$, and that $\sqrt{\frac{a^4bb}{cc}}$ is $\frac{aab}{c}$, and that $\sqrt{\frac{9aazz}{25bb}}$ is $\frac{3az}{5b}$, and that $\sqrt{\frac{4}{9}}$ is $\frac{2}{3}$, and that $\sqrt[3]{\frac{8b^6}{27a^3}}$ is $\frac{2bb}{3a}$, and that $\sqrt[4]{aabb}$ is $\sqrt{ab}$. Moreover, that $b\sqrt{aacc}$, or b into $\sqrt{aacc}$, is b into ac or abc. And that $3c\sqrt{\frac{9aazz}{25bb}}$ is $3c \times \frac{3az}{5b}$, or $\frac{9acz}{5b}$. And that $\frac{a+3x}{c}\sqrt{\frac{4bbx^4}{81aa}}$ is $\frac{a+3x}{c} \times \frac{2bxx}{9a}$, or $\frac{2abxx+6bx^3}{9ac}$.

I say, these are all evident, because it will appear, at first Sight, that the propos'd Quantities are produc'd by multiplying the Roots into themselves (as aa from $a \times a$, $aacc$ from ac into ac, $9aacc$ from $3ac$ into $3ac$, &c.) But when Quantities consist of several Terms, the Business is perform'd as in Numbers. Thus, to extract the Square Root out of $aa+2ab+bb$, in the first Place, write the Root of the first Term aa, viz. a in the Quotient, and having subtracted its Square $a \times a$, there will remain $2ab+bb$ to find the Remainder of the Root by. Say therefore, How many times is the double of the Quotient, or $2a$, contain'd in the first Term of the Remainder $2ab$? I answer b [times], therefore write b in the Quotient, and having subtracted the Product of b into $\overline{2a+b}$, or $2ab+bb$, there will remain nothing. Which shews that the Work is finish'd, the Root coming out $a+b$.

$$\begin{array}{l} aa+2ab+bb\ (a+b \\ \underline{aa} \\ 0\ +2ab+bb \\ \underline{\quad +2ab+bb} \\ \qquad 0 \qquad 0 \end{array}$$

And thus, to extract the Root out of $a^4+6a^3b+5a^2bb-12ab^3+4b^4$, first, set in the Quotient the Root of the first Term a^4, viz. aa, and having subtracted its Square $aa \times aa$, or a^4, there will remain $6a^3b+5aabb-12ab^3+4b^4$ to find the Remainder of the Root. Say therefore, How many times is $2aa$ contain'd in $6a^3b$? Answer $3ab$; wherefore write $3ab$ in the Quotient, and having subtracted the Product of $3ab$ into $\overline{2aa+3ab}$, or $6a^3b+9aabb$, there will yet remain $-4aabb-12ab^3+4b^4$ to carry on the Work. Therefore say again, How

many

many times is the double of the Quotient, *viz.* $2aa + 6ab$ contain'd in $-4aabb - 12ab^3$, or, which is the same Thing, say, How many times is the double of the first Term of the Quotient, or $2aa$, contain'd in the first Term of the Remainder $-4aabb$? Answer $-2bb$. Then having writ $-2bb$ in the Quotient, and subtracted the Product $-2bb$ into $2aa + 6ab - 2bb$, or $-4aabb - 12ab^3 + 4b^4$, there will remain nothing. Whence it follows, that the Root is $aa + 3ab - 2bb$.

$$
\begin{array}{l}
a^4 + 6a^3b + 5aabb - 12ab^3 + 4b^4 \ (aa + 3ab - 2bb \\
\underline{a^4} \\
0. + 6a^3b + 5aabb - 12ab^3 + 4b^4 \\
\quad\ \underline{6a^3b + 9aabb} \\
\quad\ 0 \quad -4aabb - 12ab^3 + 4b^4 \\
\qquad\quad \underline{-4aabb - 12ab^3 + 4b^4} \\
\qquad\qquad 0 \qquad 0 \qquad 0
\end{array}
$$

And thus the Root of the Quantity $xx - ax + \frac{1}{4}aa$ is $x - \frac{1}{2}a$; and the Root of the Quantity $y^4 + 4y^3 - 8y + 4$ is $yy + 2y - 2$; and the Root of the Quantity $16a^4 - 24aaxx + 9x^4 + 12bbxx - 16aabb + 4b^4$ is $3xx - 4aa + 2bb$, as may appear by the Diagrams underneath:

$$
\begin{array}{l}
xx - ax + \frac{1}{4}aa \ (x - \frac{1}{2}a \\
\underline{xx} \\
\underline{0 - ax + \frac{1}{4}aa} \\
\quad 0 \quad\ 0
\end{array}
$$

$$
\begin{array}{l}
9x^4 \begin{array}{l} -24aa \\ +12bb \end{array} x^2 \begin{array}{l} +16a^4 \\ -16aab^2 \\ +4b^4 \end{array} \left(3x^2 \begin{array}{l} -4aa \\ +2bb \end{array} \right. \\
\underline{9x^4} \\
0 \quad \begin{array}{l} -24aa \\ +12bb \end{array} x^2 \begin{array}{l} +16a^4 \\ -16a^2b^2 \\ +4b^4 \end{array} \\
\qquad\quad 0 \qquad\qquad 0
\end{array}
$$

$$
\begin{array}{l}
y^4 + 4y^3 * - 8y + 4\ (yy + 2y - 2 \\
\underline{y^4} \\
0 \quad \underline{4y^3 \quad + 4yy} \\
\quad\ 0 \quad\ - 4yy \\
\qquad\quad \underline{- 4yy - 8y + 4} \\
\qquad\qquad 0 \quad 0 \quad 0
\end{array}
$$

If you would extract the Cube Root of $a^3 + 3aab + 3abb + b^3$, the Operation is [perform'd] thus:

$$
\begin{array}{l}
a^3 + 3aab + 3abb + b^3\ (a + b \\
\underline{a^3} \\
3aa)\ \underline{0 + 3aab\ (b \qquad\qquad} \\
\quad a^3 + 3aab + 3abb + b^3 \\
\quad 0 \qquad 0 \qquad 0 \qquad 0
\end{array}
$$

Extract first the Cube Root of the first Term a^3, viz. a, and set it down in the Quotient: Then, subtracting its Cube a^3, say, How many times is its triple Square, or $3aa$, contain'd in the next Term of the Remainder $3aab$? and there comes out b; wherefore write b in the Quotient, and subtracting the Cube of the Quotient, there will remain o. Therefore $a + b$ is the Root.

After the same Manner, if the Cube Root is to be extracted out of $z^6 + 6z^5 - 40z^3 + 96z - 64$, it will come out $zz + 2z - 4$. And so in higher Roots.

Of the REDUCTION *of* FRACTIONS *and* RADICAL [*Quantities.*]

THE Reduction of Fractions and Radical Quantities is of Use in the preceding Operations, and is [of reducing them] either to the least Terms, or to the same Denomination.

Of

Of the REDUCTION *of* FRACTIONS *to the least Terms.*

FRACTIONS are reduc'd to the least Terms by dividing the Numerators and Denominators by the greatest common Divisor. Thus the Fraction $\frac{aac}{bc}$ is reduc'd to a more Simple one $\frac{aa}{b}$ by dividing both aac and bc by c; and $\frac{203}{667}$ is reduc'd to a more Simple one $\frac{7}{23}$ by dividing both 203 and 667 by 29; and $\frac{203aac}{667bc}$ is reduc'd to $\frac{7aa}{23b}$ by dividing by $29c$. And so $\frac{6a^3 - 9acc}{6aa + 3ac}$ becomes $\frac{2aa - 3cc}{2a + c}$ by dividing by $3a$. And $\frac{a^3 - aab + abb - b^3}{aa - ab}$ becomes $\frac{aa + bb}{a}$ by dividing by $a - b$.

And after this Method, the Terms after Multiplication or Division may be for the most part abridg'd. As if you were to multiply $\frac{2ab^3}{3ccd}$ by $\frac{9acc}{bdd}$, or divide it by $\frac{bdd}{9acc}$, there will come out $\frac{18aab^3cc}{3bccd^3}$, and by Reduction $\frac{6aabb}{d^3}$. But in these Cases, it is better to abbreviate the Terms before the Operation, by dividing those Terms [first] by the greatest common Divisor, which you would be oblig'd to do afterwards. Thus, in the Example before us, if I divide $2ab^3$ and bdd by the common Divisor b, and $3ccd$ and $9acc$ by the common Divisor $3cc$, there will come out the Fraction $\frac{2abb}{d}$ to be multiply'd by $\frac{3a}{dd}$, or to be divided by $\frac{dd}{3a}$, there coming out $\frac{6aabb}{d^3}$ as above. And so $\frac{aa}{c}$ into $\frac{c}{b}$ becomes $\frac{aa}{1}$ into $\frac{1}{b}$, or $\frac{aa}{b}$. And $\frac{aa}{c}$ divided by $\frac{b}{c}$ becomes

becomes aa divided by b, or $\frac{aa}{b}$. And $\frac{a^3 - axx}{xx}$ into $\frac{cx}{aa + ax}$ becomes $\frac{a - x}{x}$, into $\frac{c}{1}$, or $\frac{ac}{x} - c$. And 28 divided by $\frac{7}{3}$ becomes 4 divided by $\frac{1}{3}$, or 12.

Of the Invention of Divisors.

TO this Head may be referr'd the Invention of Divisors, by which any Quantity may be divided. If it be a simple Quantity, divide it by its least Divisor, and the Quotient by its least Divisor, till there remain an indivisible Quotient, and you will have all the prime Divisors of [that] Quantity. Then multiply together each Pair of these Divisors, each ternary [or three] of them, each quaternary, &c. and you will also have all the compounded Divisors. As, if all the Divisors of the Number 60 are requir'd, divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and there will remain the indivisible Quotient 5. Therefore the prime Divisors are 1, 2, 2, 3, 5; those compos'd of the Pairs 4, 6, 10, 15; of the Ternaries 12, 20, 30; and of all of them 60. Again, If all the Divisors of the Quantity $21abb$ are desir'd, divide it by 3, and the Quotient $7abb$ by 7, and the Quotient abb by a, and the Quotient bb by b, and there will remain the prime Quotient b. Therefore the prime Divisors are 1, 3, 7, a, b, b; and those compos'd of the Pairs 21, $3a$, $3b$, $7a$, $7b$, ab, bb; those compos'd of the Ternaries $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb; and those of the Quaternaries $21ab$, $21bb$, $3abb$, $7abb$; that of the Quinaries $21abb$. After the same Way all the Divisors of $2abb - 6aac$ are 1, 2, a, $bb - 3ac$, $2a$, $2bb - 6ac$, $abb - 3aac$, $2abb - 6aac$.

If after a Quantity is divided by all its simple Divisors, it remains [still] compounded, and you suspect it has some compounded Divisor, [order it or] dispose it according to the Dimensions of any of the Letters in it, and in the Room of that Letter substitute successively three or more Terms of this Arithmetical Progression, *viz.* 3, 2, 1, 0, −1, −2, and set the resulting Terms together with all their Divisors, by the corresponding Terms of the Progression, setting down also the Signs of the Divisors, both Affirmative and

and Negative. Then set also down the Arithmetical Progressions which run thro' the Divisors of all the Numbers proceeding from the greater Terms to the less, in the Order that the Terms of the Progression $3, 2, 1, 0, -1, -2$, proceed, and whose Terms differ either by Unity, or by some Number which divides the highest Term of the Quantity propos'd. If any Progression of this kind occurs, that Term of it which stands in the same Line with the Term 0 of the first Progression, divided by the Difference of the Terms, will compose the Quantity by which you are to attempt the Division.

As if the Quantity be $x^3 - xx - 10x + 6$, by substituting, one by one, the Terms of this Progression $1.\ 0.\ -1$, for x, there will arise the Numbers $-4, 6, +14$, which, together with all their Divisors, I place right against the Terms of the Progression $1.\ 0.\ -1$. after this Manner:

1	4	1. 2. 4.	+ 4.
0	6	1. 2. 3. 6	+ 3.
−1	14	1. 2. 7. 14	+ 2.

Then, because the highest Term x^3 is divisible by no Number but Unity, I seek among the Divisors a Progression whose Terms differ by Unity, and (proceeding from the highest to the lowest) decrease as the Terms of the lateral Progression $1.\ 0.\ -1$. And I find only one Progression of this Sort, *viz.* 4. 3. 2. whose Term therefore $+3$ I chuse, which stands in the same Line with the Term 0 of the first Progression $1.\ 0.\ -1$. and I attempt the Division by $x + 3$, and [find] it succeeds, there coming out $xx - 4x + 2$.

Again, if the Quantity be $6y^4 - y^3 - 21yy + 3y + 20$, for y I substitute successively $1.\ 0.\ -1$. and the resulting Numbers 7. 20. 9. with all their Divisors, I place by them as follows:

1	7	1. 7.	7.
0	20	1. 2. 4. 5. 10. 20	4.
−1	9	1. 3. 9	1.

And among the Divisors I perceive there is this decreasing Arithmetical Progression 7. 4. 1. The Difference of the Terms of this Progression, *viz.* 3. divides the highest Term of the Quantity $6y^4$. Wherefore I adjoin the Term $+4$, which

which ſtands [in the Row] oppoſite to the Term 0, divided by the Difference of the Terms, *viz.* 3, and I attempt the Diviſion by $y + \frac{4}{3}$, or, which is the ſame Thing, by $3y + 4$, and the Buſineſs ſucceeds, there coming out $2y^3 - 3yy - 3y + 5$.

And ſo, if the Quantity be $24a^5 - 50a^4 + 49a^3 - 140a^2 + 64a + 30$, the Operation will be as follows:

2	42	1.2.3.6.7.14.21.42	+3. +3. +7.
1	23	1.23.	+1. −1. +1.
0	30	1.2.3.5.6.10.15.30	−1. −5. −5.
−1	297	1.3.9.11.27.33.99.297	−3. −9. −11.

Here are three Progreſſions, whoſe Terms $-1. -5. -5$, divided by the Differences of the Terms 2, 4, 6, give three Diviſors to be try'd $a - \frac{1}{2}$, $a - \frac{5}{4}$, and $a - \frac{5}{6}$. And the Diviſion by the laſt Diviſor $a - \frac{5}{6}$, or $6a - 5$, ſucceeds, there coming out $4a^4 - 5a^3 + 4aa - 20a - 6$.

If no Diviſor occur by this Method, or none that divides the Quantity propos'd, we are to conclude, that that Quantity does not admit a Diviſor of one Dimenſion. But perhaps it may, if it be a Quantity of more than three Dimenſions, admit a Diviſor of two Dimenſions. And if ſo, that Diviſor will be found by this Method. Subſtitute in that Quantity for the Letter [or Species] as before, four or more Terms of this Progreſſion 3, 2, 1, 0. −1. −2. −3. Add and ſubtract ſingly all the Diviſors of the Numbers that reſult to or from the Squares of the correſpondent Terms of that Progreſſion, multiply'd into ſome Numeral Diviſor of the higheſt Term of the Quantity propos'd, and place right againſt the Progreſſion the Sums and Differences. Then note all the collateral Progreſſions which run thro' thoſe Sums and Difference. Then ſuppoſe $\mp C$ to be a Term of ſuch a prime Progreſſion, and $\mp B$ the Difference which ariſes by ſubducting $\mp C$ from the next ſuperior Term which ſtands againſt the Term 1 of the firſt Progreſſion, and A to be the aforeſaid Numeral Diviſor of the higheſt Term, and l [to be] a Letter which is in the propos'd Quantity, then $All \pm Bl \pm C$ will be the Diviſor to be try'd.

Thus ſuppoſe the propos'd Quantity to be $x^4 - x^3 - 5xx + 12x - 6$, for x I write ſucceſſively 3, 2, 1, 0, −1, and the Numbers that come out 39. 6. 1. −6. −21. −26. I diſpoſe [or place] together with their Diviſors in another Column in the ſame Line with them, and I add and ſubtract the

Diviſors

Divisors to and from the Squares of the Terms of the first Progression, multiply'd by the Numeral Divisor of the Term x^4, which is Unity, *viz.* to and from the Terms 9.4.1.0.1.4, and I dispose likewise the Sums and Differences on the Side. Then I write, as follows, the Progressions which occur among the same. Then I make Use of the Terms of these Progressions 2 and —3, which stand opposite to the Term 0 in that Progression which is in the first Column, successively

3	39	1.3.13.39	9	—30.—4.6.8.10.12.22.48	—4. 6
2	6	1.2. 3. 6	4	—2.1.2.3.5.6.7.10.	—2. 3
1	1	1.	1	0.2.	0. 0
0	6	1.2. 3. 6	0	—6.—3.—2.—1.1.2.3.6	2—3
—1	21	1.3. 7.21	1	—20.—6.—2 0.2.4.8.22	4—6
—2	26	1.2.13.26	4	—22.—9.2.3.5.6.17.30	6—9

for $\mp$ C, and I make Use of the Differences that arise by subtracting these Terms from the superior Terms 0 and 0, *viz.* —2 and +3 respectively for $\mp$ B. Also Unity for A; and x for l. And so in the Room of A $ll \pm$ B $l \pm$ C, I have these two Divisors to try, *viz.* $xx+2x-2$, and $xx-3x+3$, by both of which the Business succeeds.

Again, if the Quantity $3y^5-6y^4+y^3-8yy-14y+14$ be propos'd, the Operation will be as follows: First, I attempt the Business by adding and subtracting to and from the Squares of the Terms of the Progression 1.0.—1, making Use of 1 first, but the Business does not succeed. Where-

3	170		27		—7. 17
2	38	1.2.19.38	12	—16 —7.10.11.13.14.31.50	—7.—11
1	10	1.2. 5.10	3	—7.—2.1.2.4.5.8.13	—7. 5
0	14	1.2. 7.14	0	—14—7.—2.—1.1.2.7.14	—7.—1
—1	10	1.2. 5.10	3	—7.—2.1.2.4.5.8.13	—7.—7
—2	190		12		—7.—13

fore, in the room of A, I make Use of 3, the other Divisor of the highest Term; and these Squares being multiply'd by 3, I add and subtract the Divisors to and from the Products, *viz.* 12.3.0.3, and I find these two Progressions in the resulting Terms, —7.—7.—7.—7, and 11.5.—1.—7. For Expedition sake, I had neglected the Divisors of the outermost Terms 170 and 190. Wherefore, the Progressions being continu'd upwards and downwards, I take the next Terms, *viz.* —7 and 17 at the Top, and —7 and —13 at Bottom, and I try if these being subducted from the Num-

bers

bers 27 and 12, which stand against them in the 4th Column, [their] Differences divide those [Numbers] 170 and 190, which stand against them in the second Column. And the Difference between 27 and -7, that is, 34, divides 170; and the Difference of 12 and -7, that is, 19, divides 190. Also the Difference between 12 and 13, that is, 10, divides 170, but the Difference between 27 and 17, that is, 25, does not divide 190. Wherefore I reject the latter Progression. According to the former, $\mp C$ is -7, and $\mp B$ is nothing; the Terms of the Progression having no Difference. Wherefore the Divisor to be try'd $All \pm Bl \pm C$ will be $3yy+7$. And the Division succeeds, there coming out $y^3-2yy-2y+2$.

If after this Way, there can be found no Divisor which succeeds, we are to conclude, that the propos'd Quantity will not admit of a Divisor of two Dimensions. The same Method may be extended to the Invention of Divisors of more Dimensions, by seeking in the aforesaid Terms and Differences, not Arithmetical Progressions, but some others, the first, second, and third Differences of whose Terms are in Arithmetical Progression: But the Learner ought not to be detain'd about them.

Where there are two Letters in the propos'd Quantity, and all its Terms ascend to equally high Dimensions; put Unity for one of those Letters, then, by the preceding Rules, seek a Divisor, and compleat the deficient Dimensions of this Divisor, by restoring that Letter for Unity. As if the Quantity be $6y^4-cy^3-21ccyy+3c^3y+20c^4$, where all the Terms are of four Dimensions, for c I put 1, and the Quantity becomes $6y^4-y^3-21yy+3y+20$, whose Divisor, as above, is $3y+4$; and having compleated the deficient Dimension of the last Term by a [correspondent] Dimension of c, you have $3y+4c$ [for] the Divisor sought. So, if the Quantity be $x^4-bx^3-5bbxx+12b^3x-6b^4$, putting 1 for b, and having found $xx+2x-2$ the Divisor of the resulting Quantity $x^4-x^3-5xx+12x-6$, I compleat its deficient Dimensions by [respective] Dimensions of b, and so I have $xx+2bx+2bb$ the Divisor sought.

Where there are three or more Letters in the Quantity propos'd, and all its Terms ascend to the same Dimensions, the Divisor may be found by the precedent Rules; but more expeditiously after this Way: Seek all the Divisors of all the Terms in which some [one] of the Letters is not,

not, and also of all the Terms in which some other of the Letters is not; as also of all the Terms in which a third, fourth, and fifth Letter is not, if there are so many Letters; and so run over all the Letters: And in the same Line with those Letters place the Divisors respectively. Then see if in any Series of Divisors going through all the Letters, all the Parts involving, only one Letter can be as often found as there are Letters (excepting only one) in the Quantity propos'd; and [likewise] if the Parts involving two Letters [may be found] as often as there are Letters (excepting two) in the Quantity propos'd. If so, all those Parts taken together under their [proper] Signs will be the Divisor sought.

As if there were propos'd the Quantity $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$; the Divisors of one Dimension of the Terms $8b^3 - 12bbc - 4bcc + 6c^3$, in which x is not (found out by the preceding Rules) will be $2b - 3c$, and $4b - 6c$; and of the Terms $12x^3 + 9cxx + 8ccx + 6c^3$, in which b is not, there will be only one Divisor $4x + 3c$; and of the Terms $12x^3 - 14bxx - 12bbx + 8b^3$, in which there is not c, there will be the Divisors $2x - b$ and $4x - 2b$. I dispose these Divisors in the same Lines with the Letters x, b, c, as you here see;

$$\begin{array}{l|l} x & 2b - 3c.\ \ 4b - 6c. \\ b & 4x + 3c. \\ c & 2x - b.\ \ 4x - 2b. \end{array}$$

Since there are three Letters, and each of the Parts of the Divisors only involve one of the Letters, those Parts ought to be found twice in the Series of Divisors. But the Parts $4b$, $6c$, $2x$, b of the Divisors $4b - 6c$ and $2x - b$, only occur once, and are not found any where out of those Divisors whereof they are Parts. Wherefore I neglect those Divisors. There remain only three Divisors $2b - 3c$, $4x + 3c$, and $4x - 2b$. These are in the Series going through all the Letters x, b, c, and each of the Parts $2b$, $3c$, $4x$, are found in them twice as ought to be, and that with the same Signs, if only the Signs of the Divisor $2b - 3c$ be chang'd, and in its place you write $-2b + 3c$. For you may change the Signs of any Divisor. I take therefore all the Parts of these, *viz.* $2b$, $3c$, $4x$ once [apiece] under their [proper] Signs, and the Aggregate $-2b + 3c + 4x$ will be be the Divisor which was to be found. For if by this you divide the propos'd Quantity, there will come out $3xx - 2bx + 2cc - 4bb$.

Again, if the Quantity be $12x^5 - 10ax^4 - 9bx^4 - 26a^2x^3 + 12abx^3 + 6bbx^3 + 24a^3xx - 8aabxx - 8abbxx - 24b^3xx - 4a^3bx + 6aabbx - 12ab^3x + 18b^4x + 12a^4b + 32aab^3 - 12b^5$, I place the Divisors of the Terms in which x is not, by x; and those Terms in which a is not, by a; and those in which b is not, by b, as you here see. Then I perceive that all those that

x	$b.\ 2b.\ 4b.\ aa + 3bb.\ 2aa + 6bb.\ 4aa + 12bb.$ $bb - 3aa.\ 2bb - 6aa.\ 4bb - 12aa.$
a	$4xx - 3bx + 2bb.\ 12xx - 9bx + 6bb.$
b	$x.\ 2x.\ 3x - 4a.\ 6x - 8a.\ 3xx - 4ax.\ 6xx - 8ax.$ $2xx + ax - 3aa.\ 4xx + 2ax - 6aa.$

are but of one Dimension are to be rejected, because the Simple ones, $b.\ 2b.\ 4b.\ x.\ 2x$, and the Parts of the compounded ones, $3x - 4a.\ 6x - 8a$, are found but once in all the Divisors; but there are three Letters in the propos'd Quantity, and those Parts involve but one, and so ought to be found twice. In like Manner, the Divisors of two Dimensions, $aa + 3bb.\ 2aa + 6bb.\ 4aa + 12bb.\ bb - 3aa.$ and $4bb - 12aa$ I reject, because their Parts $aa.\ 2aa.\ 4aa.\ bb.$ and $4bb.$ involving only one Letter a or b, are not found more than once. But the Parts $2bb$ and $6aa$ of the Divisor $2bb - 6aa$, which is the only remaining one in the Line with x, and which likewise involve only one Letter, are found again [or twice], *viz.* the Part $2bb$ in the Divisor $4xx - 3bx + 2bb$, and the Part $6aa$ in the Divisor $4xx + 2ax - 6aa$. Moreover, these three Divisors are in a Series standing in the same Lines with the three Letters x, a, b; and all their Parts $2bb$, $6aa$, $4xx$, which involve only one Letter, are found twice in them, and that under their proper Signs; but the Parts $3bx$, $2ax$, which involve two Letters, occur but once in them. Wherefore, all the divers Parts of these three Divisors, $2bb$, $6aa$, $4xx$, $3bx$, $2ax$, connected under their proper Signs, will make the Divisors sought, *viz.* $2bb - 6aa + 4xx - 3bx + 2ax$. I therefore divide the Quantity propos'd by this [Divisor] and there arises $3x^3 - 4axx - 2aab - 6b^3$.

If

If all the Terms of any Quantity are not equally high, the deficient Dimensions must be fill'd up by the Dimensions of any assum'd Letter; then having found a Divisor by the precedent Rules, the assum'd Letter is to be blotted out. As if the Quantity be $12x^3 - 14bxx + 9xx - 12bbx - 6bx + 8x + 8b^3 - 12b^2 - 4b + 6$; assume any Letter, as c, and fill up the Dimensions of the Quantity propos'd by its Dimensions, after this Manner, $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$. Then having found out its Divisor $4x - 2b + 3c$, blot out c, and you'll have the Divisor requir'd, *viz.* $4x - 2b + 3$.

Sometimes Divisors may be found more easily than by these Rules. As if some Letter in the propos'd Quantity be of only one Dimension, you may seek for the greatest common Divisor of the Terms in which that Letter is found, and of the remaining Terms in which it is not found; for that Divisor will divide the whole. And if there is no such common Divisor, there will be no Divisor of the whole. For Example, if there be propos'd the Quantity $x^4 - 3ax^3 - 8aaxx + 18a^3x - cx^3 + acxx + 8aacx - 6a^3c - 8a^4$, let there be sought the common Divisor of the Terms $-cx^3 + acxx + 8aacx - 6a^3c$, in which c is only of one Dimension, and of the remaining Terms $x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4$, and that Divisor, *viz.* $xx + 2ax - 2aa$, will divide the whole Quantity.

But the greatest common Divisor of two Numbers, if it is not known [or does not appear] at first Sight, it is found by a perpetual Subtraction of the less from the greater, and of the Remainder from the [last Quantity] subtracted; and that will be the sought Divisor, which leaves nothing. Thus, to find the greatest common Divisor of the Numbers 203 and 667, subtract thrice 203 from 667, and the Remainder 58 thrice from 203, and the Remainder 29 twice from 58, and there will remain nothing; which shews, that 29 is the Divisor sought.

After the same Manner the common Divisor in Species, when it is compounded, is found, by subtracting either Quantity, or its Multiple, from the other; if those Quantities and the Remainder be order'd [or rang'd] according to the Dimensions of any Letter, as is shewn in Division, and be each Time manag'd by dividing them by all their Divisors, which are either Simple, or divide each of its

Terms

Terms as if it were a Simple one. Thus, to find the greatest common Divisor of the Numerator and Denominator of this Fraction $\frac{x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4}{x^3 - axx - 8aax + 6a^3}$, multiply the Denominator by x, that its first Term may become the same with the first Term of the Numerator. Then subtract it, and there will remain $-2ax^3 + 12a^3x - 8a^4$, which being rightly order'd by dividing by $-2a$, it becomes $x^3 - 6a^2x + 4a^3$. Subtract this from the Denominator, and there will remain $-axx - 2aax + 2a^3$; which again divided by $-a$ becomes $xx + 2ax - 2aa$. Multiply this by x, that its first Term may become the same with the first Term of the last subtracted Quantity $x^3 - 6aax + 4a^3$, from which it is to be [likewise] subtracted, and there will remain $-2axx - 4aax + 4a^3$, which divided by $-2a$, becomes also $xx + 2ax - 2aa$. And since this is the same with the former Remainder, and consequently being subtracted from it, will leave nothing, it will be the Divisor sought; by which the propos'd Fraction, by dividing both the Numerator and Denominator by it, may be reduc'd to a more Simple one, *viz.* to $\frac{xx - 5ax + 4aa}{x - 3a}$.

And so, if you have the Fraction $\frac{6a^5 + 15a^4b - 4a^3cc - 10aabcc}{9a^3b - 27aabc - 6abcc + 18bc^3}$, its Terms must be first abbreviated, by dividing the Numerator by aa, and the Denominator by $3b$: Then subtracting twice $3a^3 - 9aac - 2acc + 6c^3$ from $6a^3 + 15aab - 4acc - 10bcc$, there will remain $\begin{matrix} 15b \\ +18c \end{matrix}\, aa \begin{matrix} -10bcc \\ -12c^3 \end{matrix}$. Which being order'd, by dividing each Term by $5b + 6c$ after the same Way as if $5b + 6c$ was a simple Quantity, it becomes $3aa - 2cc$. This being multiply'd by a, subtract it from $3a^3 - 9aac - 2acc + 6c^3$, and there will remain $-9aac + 6c^3$, which being again order'd by a Division by $-3c$, becomes also $3aa - 2cc$, as before. Wherefore $3aa - 2cc$ is the Divisor sought. Which being found, divide by it the Parts of the propos'd Fraction, and you'll have $\frac{2a^3 + 5aab}{3ab - 9bc}$.

Now,

Now, if a common Divisor cannot be found after this Way, it is certain there is none at all; unless, perhaps, it be one of the Terms that abbreviate the Numerator and Denominator of the Fraction: As, if you have the Fraction $\frac{aadd - ccdd - aacc + c^4}{4aad - 4acd - 2acc + 2c^3}$, and so dispose its Terms, according to the Dimensions of the d, that the Numerator may become $\begin{matrix} aa \\ -cc \end{matrix} dd \begin{matrix} -aacc \\ +c^4 \end{matrix}$, and the Denominator $\begin{matrix} 4aa \\ -4ac \end{matrix} d \begin{matrix} -2acc \\ +2c^3 \end{matrix}$. This must first be abbreviated, by dividing each Term of the Numerator by $aa - cc$, and each of the Denominator by $2a - 2c$, just as if $aa - cc$ and $2a - 2c$ were simple Quantities; and so, in Room of the Numerator there will come out $dd - cc$, and in Room of the Denominator $2ad - cc$, from which, thus prepar'd, no common Divisor can be obtain'd. But, out of the Terms $aa - cc$ and $2a - 2c$, by which both the Numerator and Denominator are abbreviated, there comes out a Divisor, *viz.* $a - c$, by which the Fraction may be reduc'd to this, *viz.* $\frac{add + cdd - acc - c^3}{4ad - 2cc}$. Now, if neither the Terms $aa - cc$ and $2a - 2c$ had not had a common Divisor, the propos'd Fraction would have been irreducible.

And this is a general Method of finding common Divisors; but most commonly they are more expeditiously found by seeking all the prime Divisors of either of the Quantities, that is, such as cannot be divided by others, and then by trying if any of them will divide the other without a Remainder. Thus, to reduce $\frac{a^3 - aab + abb - b^3}{aa - ab}$ to the least Terms, you must find the Divisors of the Quantity $aa - ab$, viz. a and $a - b$; then you must try whether either a, or $a - b$, will also divide $a^3 - aab + abb - b^3$ without any Remainder.

Of

Of the REDUCTION *of* FRACTIONS *to a common Denominator.*

FRACTIONS are reduc'd to a common Denominator by multiplying the Terms of each by the Denominator of the other. Thus, having $\frac{a}{b}$ and $\frac{c}{d}$, multiply the Terms of one $\frac{a}{b}$ by d, and alſo the Terms of the other $\frac{c}{d}$ by b, and they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$, whereof the common Denominator is bd. And thus a and $\frac{ab}{c}$, or $\frac{a}{1}$ and $\frac{ab}{c}$ become $\frac{ac}{c}$ and $\frac{ab}{c}$. But where the Denominators have a common Diviſor, it is ſufficient to multiply them alternately by the Quotients. Thus the Fraction $\frac{a^3}{bc}$ and $\frac{a^3}{bd}$ are reduc'd to theſe $\frac{a^3d}{bcd}$ and $\frac{a^3c}{bcd}$, by multiplying alternately by the Quotients c and d, ariſing by the Diviſion of the Denominators by the common Diviſor b.

This Reduction is moſtly of Uſe in the Addition and Subſtraction of Fractions, which, if they have different Denominators, muſt be firſt reduc'd to the ſame [Denominator] before they can be added. Thus $\frac{a}{b}+\frac{c}{d}$ by Reduction becomes $\frac{ad}{bd}+\frac{bc}{bd}$, or $\frac{ad+bc}{bd}$, and $a+\frac{ab}{c}$ becomes $\frac{ac+ab}{c}$. And $\frac{a^3}{bc}-\frac{a^3}{bd}$ becomes $\frac{a^3d-a^3c}{bcd}$, or $\frac{d-c}{bcd}a^3$. And $\frac{c^4+x^4}{cc-xx}-cc-xx$ becomes $\frac{2x^4}{cc-xx}$. And ſo $\frac{2}{3}+\frac{5}{7}$ becomes $\frac{14}{21}+\frac{15}{21}$, or $\frac{14+15}{21}$, that is, $\frac{29}{21}$. And $\frac{11}{6}-\frac{3}{4}$ becomes $\frac{22}{12}-\frac{9}{12}$, or $\frac{13}{12}$. And $\frac{3}{4}-\frac{5}{12}$ becomes

becomes $\frac{9}{12} - \frac{5}{12}$, or $\frac{4}{12}$, that is $\frac{1}{3}$. And $3\frac{4}{7}$; or $\frac{3}{1} + \frac{4}{7}$ becomes $\frac{21}{7} + \frac{4}{7}$; or $\frac{25}{7}$. And $25\frac{1}{2}$ becomes $\frac{51}{2}$.

Where there are more Fractions [than two] they are to be added gradually. Thus, having $\frac{aa}{x} - a + \frac{2xx}{3a} - \frac{ax}{a-x}$; from $\frac{aa}{x}$ take a, and there will remain $\frac{aa-ax}{x}$; to this add $\frac{2xx}{3a}$, and there will come out $\frac{3a^3 - 3aax + 2x^3}{3ax}$; from whence, laſtly, take away $\frac{ax}{a-x}$, and there will remain $\frac{3a^4 - 6a^3x + 2ax^3 - 2x^4}{3aax - 3axx}$. And ſo if you have $3\frac{4}{7} - \frac{2}{3}$, firſt, you are to find the Aggregate of $3\frac{4}{7}$, *viz.* $\frac{25}{7}$, and then to take from it $\frac{2}{3}$, and there will remain $\frac{61}{21}$.

Of the REDUCTION *of* RADICAL [*Quantities*] *to their leaſt Terms.*

A Radical [Quantity,] where the Root of the whole cannot be extracted, is perform'd by extracting the Root of ſome Diviſor [of it]. Thus $\sqrt{aabc}$, by extracting the Root of the Diviſor aa, becomes $a\sqrt{bc}$. And $\sqrt{48}$, by extracting the Root of the Diviſor 16, becomes $4\sqrt{3}$. And $\sqrt{48aabc}$, by extracting the Root of the Diviſor $16aa$, becomes $4a\sqrt{3bc}$. And $\sqrt{\frac{a^3b - 4aabb + 4ab^3}{cc}}$, by extracting the Root of its Diviſor $\frac{aa - 4ab + 4bb}{cc}$, becomes $\frac{a-2b}{c}\sqrt{ab}$. And $\sqrt{\frac{aaoomm}{ppzz} + \frac{4aammmm}{pzz}}$, by extracting the

Root

Root of the Divisor $\frac{aamm}{ppzz}$, becomes $\frac{am}{pz}\sqrt{oo+4mp}$. And $6\sqrt{\frac{75}{98}}$, by extracting the Root of the Divisor $\frac{25}{49}$, becomes $\frac{30}{7}\sqrt{\frac{3}{2}}$, or $\frac{30}{7}\sqrt{\frac{6}{4}}$, and by yet extracting the Root of the Denominator, it becomes $\frac{15}{7}\sqrt{6}$. And so $a\sqrt{\frac{b}{a}}$, or $a\sqrt{\frac{ab}{aa}}$, by extracting the Root of the Denominator, becomes $\sqrt{ab}$. And $\sqrt[3]{8a^3b+16a^4}$, by extracting the Cube Root of its Divisor $8a^3$, becomes $2a\sqrt[3]{b+2a}$. And not unlike [this] $\sqrt[4]{a^3x}$, by extracting the Square Root of its Divisor aa, becomes $\sqrt{a}$ into $\sqrt[4]{ax}$, or by extracting the Biquadratick Root of the Divisor a^4, it becomes $a\sqrt[4]{\frac{x}{a}}$. And so $\sqrt[6]{a^7x^5}$ is chang'd into $a\sqrt[6]{ax^5}$, or into $ax\sqrt[6]{\frac{a}{x}}$, or into $\sqrt{ax}\times\sqrt[3]{aax}$.

Moreover, this Reduction is not only of Use for abbreviating of Radical Quantities, but also for their Addition and Subtraction, if they agree in their Roots when they are reduc'd to the most simple Form; for then they may be added, which otherwise they cannot. Thus, $\sqrt{48}+\sqrt{75}$ by Reduction becomes $4\sqrt{3}+5\sqrt{3}$, that is, $9\sqrt{3}$. And $\sqrt{48}-\sqrt{\frac{16}{27}}$ by Reduction becomes $4\sqrt{3}-\frac{4}{9}\sqrt{3}$, that is, $\frac{32}{9}\sqrt{3}$. And thus, $\sqrt{\frac{4ab^3}{cc}}+\sqrt{\frac{a^3b-4aabb+4ab^3}{cc}}$, by extracting what is Rational in it, becomes $\frac{2b}{c}\sqrt{ab}+\frac{a-2b}{c}\sqrt{ab}$, that is, $\frac{a}{c}\sqrt{ab}$. And $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}$ becomes $2a\sqrt[3]{b+2a}-b\sqrt[3]{b+2a}$, that is, $2a-b\sqrt[3]{b+2a}$.

Of

Of the REDUCTION *of* RADICAL [*Quantities*] *to the same Denomination.*

WHEN you are to multiply or divide Radicals of a different Denomination, you must [first] reduce them to the same Denomination, by prefixing that Radical Sign whose Index is the least Number, which their Indices divide without a Remainder, and by multiplying the Quantities under the Signs so many times, excepting one, as that Index is become greater. For so $\sqrt{ax}\sqrt[3]{aax}$ becomes $\sqrt[6]{a^3x^3}$ into $\sqrt[6]{a^4xx}$, that is, $\sqrt[6]{a^7x^5}$. And $\sqrt{a}$ into $\sqrt[4]{ax}$ becomes $\sqrt[4]{aa}$ into $\sqrt[4]{ax}$, that is, $\sqrt[4]{a^3x}$. And $\sqrt{6}$ into $\sqrt[4]{\frac{5}{6}}$ becomes $\sqrt[4]{36}$ into $\sqrt[4]{\frac{5}{6}}$, that is, $\sqrt[4]{30}$. By the same Reason, $a\sqrt{bc}$ becomes $\sqrt{aa}$ into $\sqrt{bc}$, that is, $\sqrt{aabc}$. And $4a\sqrt{3bc}$ becomes $\sqrt{16aa}$ into $\sqrt{3bc}$, that is $\sqrt{48aabc}$. And $2a\sqrt[3]{b+2a}$ becomes $\sqrt[3]{8a^3}$ into $\sqrt[3]{b+2a}$, that is, $\sqrt[3]{8a^3b+16a^4}$. And so $\frac{\sqrt{ac}}{b}$ becomes $\frac{\sqrt{ac}}{\sqrt{bb}}$, or $\sqrt{\frac{ac}{bb}}$. And $\frac{6abb}{\sqrt{18ab^3}}$ becomes $\frac{\sqrt{36aab^4}}{\sqrt{18ab^3}}$, or $\sqrt{2ab}$. And so in others.

Of the REDUCTION *of* RADICALS *to more simple Radicals, by the Extraction of Roots.*

THE Roots of Quantities, which are compos'd of Integers and Radical Quadraticks, extract thus: Let A denote the greater Part of any Quantity, and B the lesser Part; and $\frac{A+\sqrt{AA-BB}}{2}$ will be the Square of the greater Part of the Root; and $\frac{A-\sqrt{AA-BB}}{2}$ will be the Square of the lesser Part, which is to be joyn'd to the greater Part with the Sign of B. As if the Quantity be $3+\sqrt{8}$, by writing 3 for A, and $\sqrt{8}$ for B, $\sqrt{AA-BB}=1$, and thence the Square of the greater Part of the Root $\frac{3+1}{2}$, that is, 2, and the Square of the less $\frac{3-1}{2}$, that is, 1. Therefore the Root is $1+\sqrt{2}$. Again, if you are to extract the Root of $\sqrt{32}-\sqrt{24}$, by putting $\sqrt{32}$ for A, and $\sqrt{24}$ for B, $\sqrt{AA-BB}$ will $=\sqrt{8}$, and thence $\frac{\sqrt{32}+\sqrt{8}}{2}$, and $\frac{\sqrt{32}-\sqrt{8}}{2}$, that is, $3\sqrt{2}$ and $\sqrt{2}$ will be the Squares of the Parts of the Root. The Root therefore is $\sqrt[4]{18}-\sqrt[4]{2}$. After the same manner, if, out of $aa+2x\sqrt{aa-xx}$ you are to extract the Root, for A write aa, and for B $2x\sqrt{aa-xx}$, and $AA-BB$ will $=a^4-4aaxx+4x^4$, the Root whereof is $aa-2xx$. Whence the Square of one Part of the Root will be $aa-xx$, and that of the other xx; and so the Root [will be] $x+\sqrt{aa-xx}$. Again, if you have $aa+5ax-2a\sqrt{ax+4xx}$, by writing $aa+5ax$ for A, and $2a\sqrt{ax+4xx}$ for B, $AA-BB$ will $=a^4+6a^3x+9aaxx$, whose Root is $aa+3ax$. Whence the Square of the greater Part of the Root will be $aa+4ax$, and that of the lesser Part ax, and the Root $\sqrt{aa+4ax}-\sqrt{ax}$. Lastly, if you have $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$, putting $6+\sqrt{8}=A$, and $-\sqrt{12}-\sqrt{24}=B$, $AA-BB=8$; whence the greater Part of the Root is $\sqrt{3+\sqrt{8}}$, that is as above $1+\sqrt{2}$, and the lesser Part $\sqrt{3}$, and consequently the Root it self $1+\sqrt{2}-\sqrt{3}$. But where there are more of this sort of Radical Terms, the Parts of the Root may be sooner found, by dividing the Product of any two of the Radicals by some third Radical, which [shall] produce a Rational and Integer Quotient. For the Root of that Quotient will be double of the Part of the Root sought. As in the last Example, $\frac{\sqrt{8}\times\sqrt{12}}{\sqrt{24}}=2$. $\frac{\sqrt{8}\times\sqrt{24}}{\sqrt{12}}=4$. And $\frac{\sqrt{12}\times\sqrt{24}}{\sqrt{8}}=6$. Therefore the Parts of the Root are 1, $\sqrt{2}$, $\sqrt{3}$ as above.

There

There is also a Rule of extracting higher Roots out of Numeral Quantities [consisting] of two Parts, whose Squares are commensurable. Let there be the Quantity $A \pm B$. And its greater Part A. And the Index of the Root to be extracted c. Seek the least Number N, whose Power N^c is [may be] divided by $AA - BB$, without any Remainder, and let the Quotient be Q. Compute $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$ in the nearest Integer Numbers. Let it be r. Divide $A\sqrt{Q}$ by the greatest rational Divisor. Let the Quotient be s, and let $\frac{r+\frac{n}{r}}{2s}$ in the next greatest Integers be [called] t. And $\frac{ts + \sqrt{ttss - n}}{\sqrt[2c]{Q}}$ will be the Root sought, if the Root can be extracted.

As if the Cube Root be to be extracted out of $\sqrt{968} + 25$; $AA - BB$ will $= 343$; and 7, 7, 7 will be its Divisors; therefore $N = 7$ and $Q = 1$. Moreover, $\overline{A+B} \times \sqrt{Q}$, or $\sqrt{968} + 25$, having extracted the former Part of the Root is a little greater than 56, and its Cube Root in the nearest Numbers is 4; therefore $r = 4$. Moreover, $A\sqrt{Q}$, or $\sqrt{968}$, by taking out whatever is Rational, becomes $22\sqrt{2}$. Therefore $\sqrt{2}$ its Radical Part is s, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{5\frac{3}{4}}{2\sqrt{2}}$ in the nearest Integer Numbers is 2. Therefore $t = 2$. Lastly, ts is $2\sqrt{2}$, $\sqrt{ttss - n}$ is 1, and $\sqrt[2c]{Q}$, or $\sqrt[6]{1}$, is 1. Therefore $2\sqrt{2}+1$ is the Root sought, if it can be extracted. I try therefore by Multiplication if the Cube of $2\sqrt{2}+1$ be $\sqrt{968}+25$, and it succeeds.

Again, if the Cube Root is to be extracted out of $68 - \sqrt{4374}$, $AA - BB$ will be $= 250$, whose Divisors are 5, 5, 5, 2. Therefore $N = 5 \times 2 = 10$, and $Q = 4$. And $\sqrt[q]{\overline{A+B} \times \sqrt{Q}}$, or $\sqrt[3]{\overline{68 + \sqrt{4374}} \times 2}$ in the nearest Integer Numbers is $7 = r$. Moreover, $A\sqrt{Q}$, or $68\sqrt{4}$, by ex-

extracting [or taking out] what is Rational, becomes $136\sqrt{1}$. Therefore $s=1$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{7+\frac{10}{7}}{2}$ in the nearest Integer Numbers is $4=t$. Therefore $ts=4$, $\sqrt{ttss-n}=\sqrt{6}$, and $\sqrt[2c]{Q}=\sqrt[6]{4}$, or $\sqrt[3]{2}$; and so the Root to be try'd is $\frac{4-\sqrt{6}}{\sqrt[3]{2}}$.

Again, if the fifth Root be to be extracted out of $29\sqrt{6}+41\sqrt{3}$; $AA-BB$ will be $=3$, and consequently $N=3$, $Q=81$, $r=5$, $s=\sqrt{6}$, $t=1$, $ts=\sqrt{6}$, $\sqrt{ttss-n}=\sqrt{3}$, and $\sqrt[2c]{Q}=\sqrt[10]{81}$, or $\sqrt[5]{9}$; and so the Root to be try'd is $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

But if in these Sorts of Operations, the Quantity be a Fraction, or its Parts have a common Divisor, extract separately the Roots of the Terms, and of the Factors. As if the Cube Root be to be extracted out of $\sqrt{242}-12\frac{1}{2}$, this, having reduc'd its Parts to a common Denominator, will become $\frac{\sqrt{968}-25}{2}$. Then having extracted separately the Cube Root of the Numerator and the Denominator, there will come out $\frac{2\sqrt{2}-1}{\sqrt[3]{2}}$. Again, if you are to extract any Root out of $\sqrt[3]{3993}+\sqrt[6]{17578125}$; divide the Parts by the common Divisor $\sqrt[3]{3}$, and there will come out $11+\sqrt{125}$. Whence the propos'd Quantity is $\sqrt[3]{3}$ into $11+\sqrt{125}$, whose Root will be found by extracting separately the Root of each Factor $\sqrt[3]{3}$, and $11+\sqrt{125}$.

Of

Of the Form of an ÆQUATION.

ÆQUATIONS, which are either two Ranks of Quantities, equal to one another, or one Rank taken equal to nothing, are to be consider'd chiefly after two Ways; either as the last Conclusions to which you come in the Resolution of Problems; or as Means, by the Help whereof you are to obtain [other] final Æquations. An Æquation of the former Kind is compos'd only out of one unknown Quantity involv'd with known ones, If the Problem be determin'd, and proposes something certain to be found out. But those of the latter Kind involve several unknown Quantities, which, for that Reason, must be compar'd among one another, and so connected, that out of all there may emerge a new Æquation, in which there is only one unknown Quantity which we seek; [and] that Æquation must be transform'd most commonly various Ways, untill it becomes the most Simple that it can, and also like some of the following Degrees of them, in which x denotes the Quantity sought, according to whose Dimensions the Terms, as you see, are order'd, [or rang'd] and p, q, r, s, [denote] any other Quantities from which, being known and determin'd, x is also determin'd, and may be investigated by Methods hereafter to be explain'd.

$x = p.$	Or, $x - p = 0.$
$xx = px + q.$	$xx - px - q = 0.$
$x^3 = px^2 + qx + r.$	$x^3 - pxx - qx - r = 0.$
$x^4 = px^3 + qx^2 + rx + s.$	$x^4 - px^3 - qx^2 - rx - s = 0.$
&c.	&c.

After this Manner therefore the Terms of Æquations are to be reduc'd, [or order'd] according to the Dimensions of the unknown Quantity, so that [those] may be in the first Place, in which the unknown Quantity is of the most Dimensions, as x, xx, x^3, x^4, &c. and those in the second Place, in which [x] is of the next greatest Dimension, and so on. As to what regards the Signs, they may stand any how; and one or more of the intermediate Terms may be sometimes wanting. Thus, $x^3 * - bbx + b^3 = 0$, or $x^3 = bbx - b^3$, is an Æquation of the third Degree, and

$z^4 \begin{matrix} +a \\ -b \end{matrix} z^3 * \begin{matrix} +ab^3 \\ -b^4 \end{matrix} = 0$, is an Æquation of the fourth Degree. For the Degree of an Æquation is always estimated by the greatest Dimension of the unknown Quantity, without any Regard to the known ones, or to the intermediate Terms. But by the Defect of the intermediate Terms, the Æquation is most commonly render'd much more simple, and may be sometimes depress'd to a lower Degree. For thus, $x^4 = qxx + s$ is to be reckon'd an Æquation of the second Degree, because it may be resolv'd into two Æquations of the second Degree. For, supposing $xx = y$, and y being accordingly writ for xx in that Æquation, there will come out in its stead $yy = qy + s$, an Æquation of the second Degree; by the Help whereof when y is found, the Æquation $xx = y$ also of the second Degree, will give x.

And these are the Conclusions to which Problems are to be brought. But before I go upon their Resolution, it will be necessary to shew the Methods of transforming and reducing Æquations into Order, and the Methods of finding the final Æquations. I shall comprize the Reduction of a Simple Æquation in the following Rules.

Of ordering, [or managing] &c. a Simple ÆQUATION.

RULE I. IF there are any Quantities that destroy one another, or may be joyn'd into one by Addition or Subtraction, the Terms are that Way to be diminish'd [or reduc'd]. As if you have $5b - 3a + 2x = 5a + 3x$, take from each Side $2x$, and add $3a$, and there will come out $5b = 8a + x$. And thus, $\frac{2ab + bx}{a} - 2b = a + b$, by striking out the equivalent Quantities $\frac{2ab}{a} - b = b$, becomes $\frac{bx}{a} = a$.

To this Rule may also be referr'd the Ordering [or Management] of the Terms of an Æquation, which is usually perform'd by the Transposition of the Members to the contrary Sides under the contrary Sign. As if you had the Æquation $5b = 8a + x$, you are to find x; take from each Side

Side $8a$, or, which is the ſame Thing, tranſpoſe $8a$ to the contrary Side with its Sign chang'd, and there will come out $5b - 8a = x$. After the ſame Way, if you have $aa - 3ay = ab - bb + by$, and you are to find y; tranſpoſe $-3ay$ and $ab - bb$, ſo that there may be the Terms multiply'd by y on the one Side, and the other Terms on the other Side, and there will come out $aa - ab + bb = 3ay + by$, whence you'll have y by the fifth Rule following, *viz.* by dividing each Part by $3a + b$, for there will come out $\frac{aa - ab + bb}{3a + b} = y$. And thus the Æquation $abx + a^3 - aax = abb - 2abx - x^3$, by due ordering and tranſpoſition becomes $x^3 = \begin{matrix} aa \\ -3ab \end{matrix} x \begin{matrix} -a^3 \\ +abb \end{matrix}$, or $x^3 \begin{matrix} -aa \\ +3ab \end{matrix} x \begin{matrix} +a^3 \\ -abb \end{matrix} = 0$.

RULE II. If there is any Quantity by which all the Terms of the Æquation are multiply'd, all of them muſt be divided by that Quantity; or, if all are divided by the ſame Quantity, all muſt be multiply'd by it too. Thus, having $15bb = 24ab + 3bx$, divide all the Terms by b, and you'll have $15b = 24a + 3x$; then by 3, and you'll have $5b = 8a + x$; or, having $\frac{b^3}{ac} - \frac{bbx}{cc} = \frac{xx}{c}$, multiply all by c, and there comes out $\frac{b^3}{a} - \frac{bbx}{c} = xx$.

RULE III. If there be any irreducible Fraction, in whoſe Denominator there is found the Letter [unknown], according to whoſe Dimenſions the [whole] Æquation is to be order'd [or rang'd], all the Terms of the Æquation muſt be multiply'd by that Denominator, or by ſome Diviſor of it. As if the Æquation $\frac{ax}{a - x} + b = x$ be to be order'd [or rang'd] according to x, multiply all its Terms by $a - x$ the Denominator of the Fraction $\frac{ax}{a - x}$, and there comes out $ax + ab - bx = ax - xx$, or $ab - bx = -xx$, and tranſpoſing each Part [you'll have] $xx = bx - ab$. And ſo if you have $\frac{a^3 - aab}{2cy - cc} = y - c$, and the Terms are to be order'd [or rang'd] according to [the Dimenſions of] y, multiply them by the Denominator $2cy - cc$, or, at leaſt, by its Divisor $2y - c$, that y may vanish in the Denominator, and there will come out $\frac{a^3 - abb}{c} = 2yy - 3cy + cc$, and by farther ordering $\frac{a^3 - abb}{c} - cc + 3cy = 2yy$. After the same manner $\frac{aa}{x} - a = x$, by being multiply'd by x, becomes $aa - ax = xx$, and $\frac{aabb}{cxx} = \frac{xx}{a + b - x}$, and multiplying first by xx, and then by $a + b - x$, it becomes $\frac{a^3bb + aab^3 - aabbx}{c} = x^4$.

Rule IV. If that [particular] Letter, according to whose Dimensions the Æquation is to be order'd [or rang'd], be involv'd with an irreducible Surd, all the other Terms are to be transpos'd to the other Side, their Signs being chang'd, and each Part of the Æquation must be once multiply'd by it self, if the Root be a Square one, or twice if it be a Cubick one, &c. Thus, to order the Æquation $\sqrt{aa - ax} + a = x$ according to the Letter x, transpose a to the other Side, and you have $\sqrt{aa - xx} = x - a$; and having squar'd the Parts $aa - ax = xx - 2ax + aa$, or $0 = xx - ax$, that is, $x = a$. So also $\sqrt[3]{aax + 2axx - x^3} - a + x = 0$, by transposing $-a + x$, it becomes $\sqrt[3]{aax + 2axx - x^3} = a - x$, and multiplying the Parts cubically $aax + 2axx - x^3 = a^3 - 3aax + 3axx - x^3$, or $xx = 4ax - aa$. And so $y = \sqrt{ay + yy - a\sqrt{ay - yy}}$ having squar'd the Parts, becomes $yy = ay + yy - a\sqrt{ay - yy}$, and the Terms being rightly transpos'd [it becomes] $ay = a\sqrt{ay - yy}$, or $y = \sqrt{ay - yy}$, and the Parts being again squar'd $yy = ay - yy$; and lastly, by transposing $2yy = ay$, or $2y = a$.

Rule V. The Terms, by help of the preceding Rules, being dispos'd [or rang'd] according to the Dimensions of some one of the Letters, if the highest Dimension of that Letter be multiply'd by any known Quantity, the whole Æquation must be divided by that Quantity. Thus, $2y = a$, by by dividing by 2, becomes $y=\frac{1}{2}a$. And $\frac{bx}{a}=a$, by dividing by $\frac{b}{a}$, becomes $x=\frac{aa}{b}$. And $\begin{matrix}2ac\\-cc\end{matrix}x^3\begin{matrix}+a^3\\+aac\end{matrix}xx\begin{matrix}-2a^3c\\+aacc\end{matrix}x-a^3cc=0$, by dividing by $2ac-cc$, becomes $\dfrac{\begin{matrix}2ac\\-cc\end{matrix}x^3\begin{matrix}+a^3\\+aac\end{matrix}x^2\begin{matrix}-2a^3c\\+aacc\end{matrix}x-a^3cc}{2ac-cc}=0$, or

$$x^3\frac{+a^3+aac}{2ac-cc}xx-aax-\frac{a^3c}{2a-c}=0.$$

RULE VI. Sometimes the Reduction may be perform'd by dividing the Æquation by some compounded Quantity. For thus, $y^3=\begin{matrix}-2c\\+b\end{matrix}yy+3bcy-bbc$, is reduc'd to this, *viz.* $yy=2cy+bc$, by transferring all the Terms to the same Side thus, $y^3\begin{matrix}+2c\\-b\end{matrix}yy-3bcy+bbc=0$, and dividing by $y-b$, as is shewn in the Chapter of *Division*; for there will come out $yy+2cy-bc=0$. But the Invention of this Sort of Divisors is difficult, and is more fully taught elsewhere.

RULE VII. Sometimes also the Reduction is perform'd by Extraction of the Root out of each Part of the Æquation. As if you have $xx=\frac{1}{4}aa-bb$, having extracted the Root on both Sides, there comes out $x=\sqrt{\frac{1}{4}aa-bb}$. If you have $xx+aa=2ax+bb$, transpose $2ax$ [to the other Side] and there will arise $xx-2ax+aa=bb$, and extracting the Roots of the Parts $x-a=+$, or $-b$, or $x=a\pm b$. So also having $xx=ax-bb$, add on each Side $-ax+\frac{1}{4}aa$, and there comes out $xx-ax+\frac{1}{4}aa=\frac{1}{4}aa-bb$, and extracting the Root on each Side $x-\frac{1}{2}a=\pm\sqrt{\frac{1}{4}aa-bb}$, or $x=\frac{1}{2}a\pm\sqrt{\frac{1}{4}aa-bb}$.

And thus universally if you have $xx=.px.q$, x will be $=.\frac{1}{2}p\pm\sqrt{\frac{1}{4}pp.q}$. Where $\frac{1}{2}p$ and q are to be affected with the same Signs as p and q in the former Æquation; but $\frac{1}{4}pp$ must be always made Affirmative. And this Example is a Rule according to which [or like to which] all Quadratick Æquations may be reduc'd to the Form of Simple ones. Therefore, having propos'd the Æquation $yy = \frac{2xxy}{a} + xx$, to extract the Root y, compare $\frac{2xx}{a}$ with p, that is, write $\frac{xx}{a}$ for $\frac{1}{2}p$, and $\frac{x^4}{aa} + xx$ for $\frac{1}{4}pp \cdot q$, and there will arise $y = \frac{xx}{a} + \sqrt{\frac{x^4}{aa} + xx}$, or $y = \frac{xx}{a} - \sqrt{\frac{x^4}{aa} + xx}$. After the same Way, the Æquation $yy = ay - 2cy + aa - cc$, by comparing $a - 2c$ with p, and $aa - cc$ with q, will give $y = \frac{1}{2}a - c \pm \sqrt{\frac{5}{4}aa - ac}$.

Moreover, the Biquadratick Æquation $x^4 = -aaxx + ab^3$, whose odd Terms are wanting, by help of this Rule becomes $xx = -\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}$, and extracting again the Root $x = \sqrt{-\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}}$. And so in others.

And these are the Rules for ordering one only Æquation, the Use whereof, when the Analyst is sufficiently acquainted with, so that he knows how to dispose any propos'd Æquation according to any of the Letters contain'd in it, and to obtain the Value of that Letter if it be of one Dimension, or of its greatest Power if it be of more; the Comparison of several Æquations among one another will not be difficult to him, which I am now going to shew.

Of the Transformation of two or more ÆQUATIONS *into one, in order to exterminate the unknown Quantities.*

WHEN in the Solution of any Problem, there are more Æquations than one to comprehend the State of the Question, in each of which there are several unknown Quantities; those Æquations (two by two, if there are more than two) are to be so connected, that one of the unknown Quantities may be made to vanish at each of the Operations, and so produce a new Æquation. Thus, having the Æquations $2x = y + 5$, and $x = y + 2$, by taking off equal Things out of equal Things, there will come out

$x = 3$.

$x = 3$. And you are to know, that by each Æquation one unknown Quantity may be taken away, and consequently, when there are as many Æquations as unknown Quantities, all may at length be reduc'd into one, in which there shall be only one Quantity unknown. But if there be more unknown Quantities by one than there are Æquations, then there will remain in the Æquation last resulting two unknown Quantities; and if there are more [unknown Quantities] by two than there are Æquations, then in the last resulting Æquation there will remain three; and so on.

There may also, perhaps, two or more unknown Quantities be made to vanish, by only two Æquations. As if you have $ax - by = ab - az$, and $bx + by = bb + az$; then adding Equals to Equals, there will come out $ax + bx = ab + bb$, y and z being exterminated. But such Cases either argue some Fault to lie hid in the State of the Question, or that the Calculation is erroneous, or not artificial enough. The Method by which one unknown Quantity may be [exterminated or] taken away by each of the Æquations, will appear by what follows.

The Extermination of an unknown Quantity by an Equality of its Values.

WHEN the Quantity to be exterminated is only of one Dimension in both Æquations, both its Values are to be sought by the Rules already deliver'd, and the one made equal to the other.

Thus, putting $a + x = b + y$, and $2x + y = 3b$, that y may be exterminated, the first Æquation will give $a + x - b = y$, and the second will give $3b - 2x = y$. Therefore $a + x - b = 3b - 2x$, or by [due] ordering $x = \frac{4b - a}{3}$.

And thus, $2x = y$, and $5 + x = y$, give $2x = 5 + x$, or $x = 5$.

And $ax - 2by = ab$, and $xy = bb$, give $\frac{ax - ab}{2b}$ $(= y) = \frac{bb}{x}$; and by [due] ordering [the Terms] [$xx - bx = \frac{2b^3}{a}$, or] $xx - bx - \frac{2b^3}{a} = 0$.

Also

Also $\frac{bbx - aby}{a} = ab + xy$, and $bx + \frac{ayy}{c} = 2aa$, by taking away x, give $\frac{aby + aab}{bb - ay} (= x) = \frac{2aac - ayy}{bc}$ and by Reduction $y^3 - \frac{bb}{a}yy - \frac{2aac + bbc}{a}y + bbc = 0$.

Lastly, $x + y - z = 0$, and $ay = xz$, by taking away z, give $x + y\ (= z) = \frac{ay}{x}$, or $xx + xy = ay$.

The same is also perform'd by subtracting either of the Values of the unknown Quantities from the other, and making the Remainder equal to nothing. Thus, in the first of the Examples, take away $3b - 2x$ from $a + x - b$, and there will remain $a + 3x - 4b = 0$, or $x = \frac{4b - a}{3}$.

The Extermination of an unknown Quantity by substituting its Value for it.

WHEN, at least, in one of the Æquations, the Quantity to be exterminated is only of one Dimension, its Value is to be sought in that Æquation, and then to be substituted in its Room in the other Æquation. Thus, having propos'd $xyy = b^3$, and $xx + yy = by - ax$, to exterminate x, the first will give $\frac{b^3}{yy} = x$; wherefore I substitute in the second $\frac{b^3}{yy}$ in the Room of x, and there comes out $\frac{b^6}{y^4} + yy = by - \frac{ab^3}{yy}$, and by Reduction $y^6 - by^5 + ab^3yy + b^6 = 0$.

But having propos'd $ayy + aay = z^3$, and $yz - ay = az$, to take away y, the second will give $y = \frac{az}{z - a}$. Wherefore for y I substitute $\frac{az}{z - a}$ into the first, and there comes out $\frac{a^3zz}{zz - 2az + aa} + \frac{a^3z}{z - a} = z^3$. And by Reduction, $z^4 - 2az^3 + aazz - 2a^3z + a^4 = 0$.

In

In the like manner, having propos'd $\frac{xy}{c} = z$, and $cy + zx = cc$, to take away z, I substitute in its Room $\frac{xy}{c}$ in the second Æquation, and there comes out $cy + \frac{xxy}{c} = cc$.

But a Person used to these Sorts of Computations, will oftentimes find shorter Methods [than these] by which the unknown Quantity may be exterminated. Thus, having $ax = \frac{bbx - b^3}{z}$, and $x = \frac{az}{x - b}$, if equal Quantities are multiply'd by Equals, there will come out equal Quantities, *viz.* $axx = abb$, or $x = b$.

But I leave particular Cases of this Kind to be found out by the Students as Occasion shall offer.

The Extermination of an unknown Quantity of several Dimensions in each Æquation.

WHEN the Quantity to be [exterminated or] taken away is of more than one Dimension in both the Æquations, the Value of its greatest Power must be sought in both; then, if those Powers are not the same, the Æquation that involves the lesser Power must be multiply'd by the Quantity to be taken away, or by its Square, or Cube, &c. that it may become of the same Power with the other Æquation. Then the Values of those Powers are to be made Equal, and there will come out a new Æquation, where the greatest Power or Dimension of the Quantity to be taken away is diminish'd. And by repeating this Operation, the Quantity will at length be taken away.

As if you have $xx + 5x = 3yy$, and $2xy - 3xx = 4$, to take away x, the first [Æquation] will give $xx = -5x + 3yy$, and the second $xx = \frac{2xy - 4}{3}$. I put therefore $3yy - 5x = \frac{2xy - 4}{3}$, and so x is reduc'd to only one Dimension, and so may be taken away by what I have before shewn, *viz.* by a due Reduction of the last Æquation there comes out $9yy - 15x = 2xy - 4$, or

$x =$

$x = \frac{9yy+4}{2y+15}$. I therefore substitute this Value for x in one of the Æquations first propos'd, (as in $xx+5x=3yy$) and there arises $\frac{81y^4+72yy+16}{4yy+60y+225}+\frac{45yy+20}{2y+15}=3yy$. To reduce which into Order, I multiply by $4yy+60yy+225$, and there comes out $81y^4+72yy+16+90y^3+40y+675yy+300=12y^4+180y^3+675yy$, or $69y^4-90y^3+72yy+40y+316=0$.

Moreover, if you have $y^3=xyy+3x$, and $yy=xx-xy-3$; to take away y, I multiply the latter Æquation by y, and you have $y^3=xxy-xyy-3y$, of as many Dimensions as the former. Now, by making the Values of y^3 equal to one another, I have $xyy+3x=xxy-xyy-3y$, where y is depress'd to two Dimensions. By this therefore, and the most Simple one of the Æquations first propos'd $yy=xx-xy-3$, the Quantity y may be wholly taken away by the same Method as in the former Example.

There are moreover other Methods by which this may be done, and that oftentimes more concisely. As therefore, if $yy=\frac{2x^2y}{a}+xx$, and $yy=2xy+\frac{x^4}{aa}$; that y may be extirpated, extract the Root y in each, as is shewn in the 7th Rule, and there will come out $y=\frac{xx}{a}+\sqrt{\frac{x^4}{aa}+xx}$, and $y=x+\sqrt{\frac{x^4}{aa}+xx}$. Now, by making these two Values of y equal, you'll have $\frac{xx}{a}+\sqrt{\frac{x^4}{aa}+xx}=x+\sqrt{\frac{x^4}{aa}+xx}$, and by rejecting the equal Quantities $\sqrt{\frac{x^4}{aa}+xx}$, there will remain $\frac{xx}{a}=x$, or $xx=ax$, and $x=a$.

Moreover, to take x out of the Æquations $x+y+\frac{yy}{x}=20$, and $xx+yy+\frac{y^4}{xx}=140$, take away y from the

first

first Æquation, and there remains $x + \frac{yy}{x} = 20 - y$, and squaring the Parts $xx + 2yy + \frac{y^4}{xx} = 400 - 40y + yy$, and taking away yy on both Sides, there remains $xx + yy + \frac{yyyy}{xx} = 400 - 40y$. Wherefore, since $400 - 40y$ and 140 are equal to the same Quantities, $400 - 40y$ will $=$ 140, or $y = 6\frac{1}{2}$; and so you may contract the Matter in most other Æquations.

But when the Quantity to be exterminated is of several Dimensions, sometimes there is requir'd a very laborious Calculus to exterminate it out of the Æquations; but then the Labour will be much diminish'd by the following Examples made Use of as Rules.

Rule I.

From $axx + bx + c = 0$, and $fxx + gx + h = 0$.
x being exterminated, there comes out

$$\overline{ah - bg - 2cf} \times ah + \overline{bh - cg} \times bf + \overline{agg + cff} \times c = 0.$$

Rule II.

From $ax^3 + bxx + cx + d = 0$, and $fxx + gx + h = 0$.
x being exterminated, there comes out

$$\overline{ah - bg - 2cf} \times ahh + \overline{bh - cg - 2df} \times bfh + \overline{ch - dg} \times \overline{agg + cff} + \overline{3agh + bgg + dff} \times df = 0.$$

Rule III.

From $ax^4 + bx^3 + cxx + dx + e = 0$, and $fxx + gx + h = 0$.
x being exterminated there comes out

$$\overline{ah - bg - 2cf} \times ah^3 + \overline{bh - cg - 2df} \times bfhh + \overline{agg + cff} \times \overline{chh - dgh + egg - 2efh} + \overline{3agh + bgg + dff} \times dfh + \overline{2ahh + 3bgh - dfg + eff} \times eff - \overline{bg - 2ah} \times efgg = 0.$$

RULE IV.

From $ax^3 + bxx + cx + d = 0$, and $fx^3 + gx^2 + hx + k = 0$.

x being exterminated, there comes out

$$\overline{ah - bg - 2cf} \times adhh - achk + \overline{ak + bh - cg - 2df} \times bdfh - \overline{ak + bh + 2cg + 3df} \times aakk : + \overline{cdh - ddg - cck + 2bdk} \times agg + cff : + \overline{3agh + bgg + dff - 3afk} \times adk - \overline{3ak - bh + cg + df} \times bcfk + \overline{bk - 2dg} \times bbfk : - \overline{bbk - 3adh - cdf} \times agk = 0.$$

For Example, to exterminate x out of the Æquations $xx + 5x - 3yy = 0$, and $3xx - 2xy + 4 = 0$: I respectively substitute in the first Rule for a, b, c; f, g, and h [these Quantities, *viz.*] $1, 5, -3yy$; $3, -2y$ and 4; and duly observing the Signs $+$ and $-$, there arises $\overline{4 + 10y + 18yy} \times 4 + \overline{20 - 6y^3} \times 15 + \overline{4yy - 27yy} \times -3yy = 0$, or $16 + 40y + 72yy + 300 - 90y^3 + 69y^4 = 0$.

By the like Reason that y may be expung'd out of the Æquations $y^3 - xyy - 3x = 0$, and $yy + xy - xx + 3 = 0$, I substitute into the second Rule for a, b, c, d; f, g, h, and x, [these Quantities] $1, -x; 0, -3x$; $1, x, -xx + 3$, and y respectively, and there comes out $\overline{3 - xx + xx} \times 9 - 6xx + x^4 - \overline{3x + x^3 + 6x} \times -3x + x^3 : + 3xx \times xx + 9x - 3x^3 - x^3 - 3x \times -3x = 0$. Then blotting out the superfluous Quantities and multiplying, you have $27 - 18xx + 3x^4$, $-9xx + x^6$, $+3x^4 - 18x^2 + 12x^4 = 0$. And ordering (duely) $x^6 + 18x^4 - 45xx + 27 = 0$.

Hitherto [we have discours'd] of taking away one unknown Quantity out of two Æquations. Now, if several are to be taken out of several, the Business must be done by degrees: Out of the Æquations $ax = yz$, $x + y = 2$, and $5x = y + 3z$; if the Quantity y is to be found, first, take out one of the Quantities x or z, suppose x, by substituting for it, its Value $\frac{yz}{a}$ (found by the first Æquation) in the second

cond and third Æquations; and then you will have $\frac{yz}{a} + y = z$, and $\frac{5yz}{a} = y + 3z$, out of which take away z as above.

Of the Method of taking away any Number of Surd Quantities out of Æquations.

HItherto may be referred the Extermination of Surd Quantities, by making them equal to any [other] Letters. As if you have $\sqrt{ay} - \sqrt{aa - ay} = 2a + \sqrt[3]{ayy}$, by writing t for $\sqrt{ay}$, and v for $\sqrt{aa - ay}$, and x for the $\sqrt[3]{ayy}$. you'll have the Æquations $t - v = 2a + x$, $tt = ay$, $vv = aa - ay$, and $x^3 = ayy$, out of which taking away by degrees t, v, and x, there will result an Æquation entirely free from Surdity.

How a Question may be brought to an Æquation.

AFTER the Learner has been some Time exercised in managing and transforming Æquations, Order requires that he should try his Skill in bringing Questions to an Æquation. And any Question being proposed, his Skill is particularly required to denote all its Conditions by so many Æquations. To do which, he must first consider whether the Propositions or Sentences in which it is express'd, be all of them fit to be denoted in Algebraick Terms, just as we express our Conceptions in *Latin* or *Greek* Characters. And if so, (as will happen in Questions conversant about Numbers or abstract Quantities) then let him give Names to both known and unknown Quantities, as far as Occasion requires. And the Conditions thus translated to Algebraick Terms will give as many Æquations as are necessary to solve it.

As if there are required three Numbers in continual Proportion whose Sum is 20, and the Sum of their Squares 140; putting x, y, and z for the Names of the three Numbers sought, the Question will be translated out of the Verbal to the Symbolical Expression, as follows;

The Question in Words.	*The same in Symbols.*
There are sought three Numbers on these Conditions:	x, y, z?
That they shall be continually proportional.	$x : y :: y : z$, or $xz = yy$.
That the Sum shall be 20.	$x + y + z = 20$.
And the Sum of their Squares 140.	$xx + yy + zz = 140$.

And so the Question is brought to [these] Æquations, *viz.* $xz = yy$, $x + z + y = 20$, and $xx + yy + zz = 140$, by the Help whereof x, y, and z, are to be found by the Rules deliver'd above.

But you must note, That the Solutions of Questions are (for the most part) so much the more expedite and artificial, by how fewer unknown Quantities you have at first. Thus, in the Question propos'd, putting x for the first Number, and y for the second, $\frac{yy}{x}$ will be the third Proportional; which then being put for the third Number, I bring the Question into Æquations, as follows:

The Question in Words.	*Symbolically.*
There are sought three Numbers in continual Proportion.	$(x, y, \frac{yy}{x}$?)
Whose Sum is 20.	$x + y + \frac{yy}{x} = 20$.
And the Sum of their Squares 140.	$xx + yy + \frac{y^4}{xx} = 140$.

You have therefore the Æquations $x + y + \frac{yy}{x} = 20$, and $xx + yy + \frac{y^4}{xx} = 140$, by the Reduction whereof x and y are to be determined.

Take another Example. A certain Merchant encreases his Estate yearly by a third Part, abating 100 *l* which he spends yearly in his Family; and after three Years he finds his Estate doubled, *Query*, What he is worth?

To resolve this, you must know there are [or lie hid] several Propositions, which are all thus found out and laid down.

In English.	*Algebraically.*
A Merchant has an Estate — —	x.
Out of which the first Year he expends 100 *l.*	$x - 100$.
And augments the rest by one third. —	$x - 100 + \frac{x - 100}{3}$, or $\frac{4x - 400}{3}$.
And the second Year expends 100 *l.*	$\frac{4x - 400}{3} - 100$, or $\frac{4x - 700}{3}$.
And augments the rest by a third — —	$\frac{4x - 700}{3} + \frac{4x - 700}{9}$, or $\frac{16x - 2800}{9}$.
And so the third Year expends 100 *l.* —	$\frac{16x - 2800}{9} - 100$, or $\frac{16x - 3700}{9}$.
And by the rest gains likewise one third Part	$\frac{16x - 3700}{9} + \frac{16x - 3700}{27}$, or $\frac{64x - 14800}{27}$.
And he becomes [at length] twice as rich as at first — —	$\frac{64x - 14800}{27} = 2x$.

Therefore the Question is brought to this Æquation; $\frac{64x - 14800}{27} = 2x$, by the Reduction whereof you are to find x; *viz.* Multiply it by 27, and you have $64x - 14800 = 54x$; subtract $54x$, and there remains $10x - 14800 = 0$, or $10x = 14800$, and dividing by 10, you have $x = 1480$. Wherefore, 1480 *l.* was his Estate at first, as also his Profit or Gain since.

You see therefore, that to the Solution of Questions which only regard Numbers, or the abstracted Relations of Quantities, there is scarce any Thing else required than that the Problem be translated out of the *English*, or any other Tongue it is propos'd in, into the Algebraical Language, that

is,

is, into Characters fit to denote our Conceptions of the Relations of Quantities. But it may ſometimes happen, that the Language [or the Words] wherein the State of the Queſtion is expreſs'd, may ſeem unfit to be turn'd into the Algebraical Language; but making Uſe of a few Changes, and attending to the Senſe rather than the Sound of the Words, the Verſion will become eaſy. Thus, the Forms of Speech among [ſeveral] Nations have their proper Idioms; which, where they happen, the Tranſlation out of one into another is not to be made literally, but to be determin'd by the Senſe. But that I may illuſtrate theſe Sorts of Problems, and make familiar the Method of reducing them to Æquations; and ſince Arts are more eaſily learn'd by Examples than Precepts, I have thought fit to adjoin the Solutions of the following Problems.

PROBLEM I. Having given the Sum of two Numbers (a), and the Difference of their Squares (b), to find the Numbers?

Let the leaſt of them be [call'd] x, the other will be $a-x$, and their Squares xx, and $aa-2ax+xx$ the Difference, whereof $aa-2ax$ is ſuppos'd b. Therefore, $aa-2ax=b$, and then by Reduction $aa-b=2ax$, or $\frac{aa-b}{2a}\left(=\frac{1}{2}a-\frac{b}{2a}\right)=x$. For Example, if the Sum of the Numbers, or a, be 8, and the Difference of the Squares, or b, be 16; $\frac{1}{2}a-\frac{b}{2a}$ will be $(=4-1)=3=x$, and $a-x=5$. Wherefore the Numbers are 3 and 5.

PROBLEM II. To find three Quantities, x, y, and z, the Sum of any two of which ſhall be given.

If the Sum of two of them, *viz.* x and y, be a; of x and z, b; and of y and z, c; there will be had three Æquations to determine the three Quantities ſought, x, y, and z, *viz.* $x+y=a$, $x+z=b$, and $y+z=c$. Now, that two of the unknown Quantities, *viz.* y and z *may be exterminated, take away x on both Sides in the firſt and ſecond Æquation, and you'll have $y=a-x$, and $z=b-x$, which Values ſubſtitute for y and z in the third [Æquation], and there will come out $a-x+b-x=c$, and by Reduction $x=\frac{a+b-c}{2}$; and having found x, the Æquations above $y=a-x$, and $z=b-x$, will give y and z.

EXAMPLE.

EXAMPLE. If the Sum of x and y be 9, of x and z, 10, and y and z, 13; then, in the Values of x, y, and z, write 9 for a, 10 for b, and 13 for c, and you'll have $a + b - c = 6$, and consequently x $(= \frac{a+b-c}{2}) = 3$, y $(= a - x) = 6$, and z $(= b - x) = 7$.

PROBLEM III. To divide a given Quantity into as many Parts as you please, so that the greater Parts may exceed the least by [any] given Differences.

Let (a) be a Quantity to be divided into four such Parts, and its first or least Part call x, and the Excess of the second Part above this call b, and of the third Part c, and of the fourth d; and $x + b$ will be the second Part, $x + c$ the third, and $x + d$ the fourth, the Aggregate of all which $4x + b + c + d$ is equal to the whole Line a. Take away on both Sides $b + c + d$, and there remains $4x = a - b - c - d$, or $x = \frac{a-b-c-d}{4}$.

EXAMPLE. Let there be proposed a Line of 20 Foot, so to be divided into four Parts, that the Excess of the second above the first Part shall be 2 Foot, of the third 3 Foot, and of the fourth seven Foot, and the four Parts will be x $(= \frac{a-b-c-d}{4}$, or $\frac{20-2-3-7}{4}) = 2$, $x + b = 4$, $x + c = 5$, and $x + d = 9$. After the same Manner a Quantity is divided into more Parts on the same Conditions.

PROBLEM IV. A Person being willing to distribute some Money among some Beggars, wanted eight Pence to give three Pence a peice to them; he therefore gave to each two Pence, and had three Pence remaining over and above. To find the Number of the Beggars.

Let the Number of the Beggars be x, and there will be wanting eight Pence to give all $3x$ [Number of] Pence, he has therefore $\overline{3x - 8}$ Pence; out of these he gives $2x$ Pence, and the remaining Pence $\overline{x - 8}$ are three. That is, $x - 8 = 3$, or $x = 11$.

PRO-

PROBLEM V. If two Post-Boys, *A* and *B*, at 59 Miles Distance from one another, meet in the Morning, of whom *A* rides 7 Miles in two Hours, and *B* 8 Miles in three Hours, and *B* sets out one Hour later than *A*; to find what Number of Miles *A* will ride before he meets *B*.

Call that Length x, and you'll have $59 - x$, the Length of *B*'s Journey. And since *A* travels 7 Miles in two Hours, he will make the Space x in $\frac{2x}{7}$ Hours, because 7 Miles : 2 Hours : : x Miles : $\frac{2x}{7}$ Hours. And so, since *B* rides 8 Miles in 3 Hours, he will describe his Space [or ride his Journey] $59 - x$ in $\frac{177 - 3x}{8}$ Hours. Now, since the Difference of these Times is one Hour, to the End they may become equal, add that Difference to the shorter Time $\frac{177 - 3x}{8}$, and you'll have $1 + \frac{177 - 3x}{8} = \frac{2x}{7}$, and by Reduction $35 - x$. For, multiplying by 8 you have $185 - 3x = \frac{16x}{7}$. Then multiplying also by 7 you have $1295 - 21x = 16x$, or $1295 = 37x$. And, lastly, dividing by 37, there arises $35 = x$. Therefore, 35 Miles is the Distance that *A* must ride before he meets *B*.

The same more generally.

Having given the [Velocities] Celerities [or Swiftnesses] of two moveable Bodies, *A* and *B*, tending to the same Place, together with the Interval [or Distance] of the Places and Times from and in which they begin to move; to determine the Place they shall meet in.

Suppose, the Velocity of the Body *A* to be such, that it shall pass over the Space c in the Time f; and of the Body *B* to be such as shall pass over the Space d in the Time g; and that the Interval of the Places is e, and h the Interval of the Times in which they begin to move.

CASE I. Then if both tend to the same Place, [or the same Way] and *A* be the Body that, at the Beginning of the Motion, is farthest distant from the Place they tend to: Call

call that Diſtance x, and ſubtract from it the Diſtance e, and there will remain $x-e$ for the Diſtance of B from the Place it tends to. And ſince A paſſes through the Space c in the Time f, the Time in which it will paſs over the Space x will be $\frac{fx}{c}$, becauſe the Space c is to the Time f, as the Space x to the Time $\frac{fx}{c}$. And ſo, ſince B paſſes the Space d in [the Time] g, the Time in which it will paſs the Space $x-e$ will be $\frac{gx-ge}{d}$. Now ſince the Difference of theſe Times is ſuppoſed h, that they may become equal, add h to the ſhorter Time, *viz.* to the Time $\frac{fx}{c}$ if B begins to move firſt, and you'll have $\frac{fx}{c}+h=\frac{gx-ge}{d}$, and by Reduction $\frac{cge+cdh}{cg-df}$, or $\frac{ge+dh}{g-\frac{df}{c}}=x$. But if A begins to move firſt, add h to the Time $\frac{gx-ge}{d}$, and you'll have $\frac{fx}{c}=h+\frac{gx-ge}{d}$, and by Reduction $\frac{cge-cdh}{cg-df}=x$.

Case II. If the moveable Bodies meet, and x, as before, be made the initial Diſtance of the moveable Body A, from the Place it is to move to, then $e-x$ will be the initial Diſtance of the Body B from the ſame Place; and $\frac{fx}{c}$ the Time in which A will deſcribe the Diſtance x, and $\frac{ge-gx}{d}$ the Time in which B will deſcribe its Diſtance $e-x$. To the leſſer of which Times, as above, add the Difference h, *viz.* to the Time $\frac{fx}{c}$ if B begin firſt to move, and ſo you'll have $\frac{fx}{c}+h=\frac{ge-gx}{d}$, and by Reduction $\frac{cge-cdh}{cg+df}=x$.

 Example

EXAMPLE I. If the Sun moves every Day one Degree, and the Moon thirteen, and at a certain Time the Sun be at the Beginning of *Cancer*, and, in three Days after, the Moon in the Beginning of *Aries*, the Place of their next following Conjunction is demanded. Answer in $10\frac{1}{4}$ Deg. of *Cancer*. For since they both are going towards the same Parts, and the Motion of the Moon, which is farther distant from the Conjunction, hath a later *Epocha*, the Moon will be *A*, the Sun *B*, and $\frac{cge+cdh}{cg-df}$ the Length of the Moon's Way, which, if you write 13 for c, 1 for f, d, and g, 90 for e, and 3 for h, will become $\frac{13 \times 1 \times 90 + 13 \times 1 \times 3}{13 \times 1 - 1 \times 1}$, that is, $\frac{1209}{12}$, or $100\frac{3}{4}$ Degrees; and then add these Degrees to the Beginning of *Aries*, and there will come out $10\frac{3}{4}$ Deg. of *Cancer*.

EXAMPLE II. If two Post-Boys, *A* and *B*, being in the Morning 59 Miles asunder, set out to meet each other, and *A* goes 7 Miles in 2 Hours, and *B* 8 Miles in 3 Hours, and *B* begins his Journey 1 Hour later than *A*, it is demanded how far *A* will have gone before he meets *B*. Answer, 35 Miles. For since they go towards each other, and *A* sets out first, $\frac{cge+cdh}{cg+df}$ will be the Length of his Journey; and writing 7 for c, 2 for f, 8 for d, 3 for g, 59 for e, and 1 for h, this will become $\frac{7 \times 3 \times 59 + 7 \times 8 \times 1}{7 \times 3 + 8 \times 2}$, that is, $\frac{1295}{37}$, or 35,

PROBLEM VI. Giving the Power of any Agent, to find how many such Agents will perform a given Effect a in a given Time b.

Let the Power of the Agent be such that it can produce the Effect c in the Time d, and it will be as the Time d to the Time b, so the Effect c which that Agent can produce in the Time d to the Effect which he can produce in the Time b, which then will be $\frac{bc}{d}$. Again, as the Effect of one Agent $\frac{bc}{d}$ to the Effect of all a; so that single Agent

to

to all the Agents; and thus the Number of the Agents will be $\frac{ad}{bc}$.

Example. If a Scribe can in 8 Days write 15 Sheets, how many ſuch Scribes muſt there be to write 405 Sheets in 9 Days? Anſwer 24. For if 8 be ſubſtituted for d, 15 for c, 405 for a, and 9 for b, the Number $\frac{ad}{bc}$ will become $\frac{405\times8}{9\times15}$, that is, $\frac{3240}{135}$, or 24.

Problem VII. The Forces of ſeveral Agents being given, to determine x the Time, wherein they will joyntly perform a given Effect d.

Let the Forces of the Agents A, B, C be ſuppoſed, which in the Times e, f, g can produce the Effects a, b, c reſpectively; and theſe in the Time x will produce the Effects $\frac{ax}{e}$, $\frac{bx}{f}$, $\frac{cx}{g}$; wherefore is $\frac{ax}{e}+\frac{bx}{f}+\frac{cx}{g}=d$, and by Reduction $x=\frac{d}{\frac{a}{e}+\frac{b}{f}+\frac{c}{g}}$.

Example. Three Workmen can do a Piece of Work in certain Times, *viz.* A once in 3 Weeks, B thrice in 8 Weeks, and C five times in 12 Weeks. It is deſired to know in what Time they can finiſh it joyntly? Here then are the Forces of the Agents A, B, C, which in the Times 3, 8, 12 can produce the Effects 1, 3, 5 reſpectively, and the Time is ſought wherein they can do one Effect. Wherefore, for a, b, c, d, e, f, g write 1, 3, 5, 1, 3, 8, 12, and there will ariſe $x=\frac{1}{\frac{1}{3}+\frac{3}{8}+\frac{5}{12}}$, or $\frac{8}{9}$ of a Week, that is, [allowing 6 working Days to a Week, and 12 Hours to each Day] 5 Days and 4 Hours, the Time wherein they will joyntly finiſh it.

Problem VIII. So, to compound unlike Mixtures of two or more Things, that the Things mix'd together may have a given Ratio to one another, Let the given Quantity of one Mixture be $dA + eB + fC$, the same Quantity of another Mixture $gA + hB + kC$, and the same of a third $lA + mB + nC$, where A, B, C denote the Things mix'd, and $d, e, f, g, h,$ &c. the Proportions of the same in the Mixtures. And let $pA + qB + rC$ be the Mixture which must be compos'd of the three Mixtures; and suppose x, y, and z to be the Numbers, by which if the three given Mixtures be respectively multiply'd, their Sum will become $pA + qB + rC$.

$$\text{Therefore is} \left\{ \begin{array}{l} dxA + exB + fxC \\ + gyA + hyB + kyC \\ + lzA + mzB + nzC \end{array} \right\} = pA + qB + rC.$$

And then making $dx + gy + lz = p$; $ex + hy + mz = q$, and $fx + ky + nz = r$, and by Reduction $x = \frac{p - gy - lz}{d} = \frac{r - ky - nz}{f}$. And again, the Æquations $\frac{p - gy - lz}{d} = \frac{q - hy - mz}{e}$, and $\frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}$ by Reduction give $\frac{ep - dq + dmz - elz}{eg - dh} (= y) = \frac{fq - er + enz - fmz}{fh - ek}$, which, if abbreviated by writing α for $ep - dq$, β for $dm - el$, γ for $eg - dh$, δ for $fq - er$, ζ for $en - fm$, and θ for $fh - ek$, will become $\frac{\alpha + \beta z}{\gamma} = \frac{\delta + \zeta z}{\theta}$, and by Reduction $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = z$. Having found z, put $\frac{\alpha + \beta z}{\gamma} = y$, and $\frac{p - gy - lz}{d} = x$.

Example. If there were three Mixtures of Metals melted down together; of the first of which a Pound [Averdupois] contains of Silver ℥ 12, of Brass ℥ 1, and of Tin ℥ 3; of the second, a Pound contains of Silver ℥ 1, of Brass ℥ 12, and of Tin ℥ 3; and a Pound of the third contains of Brass ℥ 14, of Tin ℥ 2, and no Silver; and let these Mixtures be so to be compounded, that a Pound of the Composition may contain of Silver ℥ 4, of Brass ℥ 9, and of Tin ℥ 3: For d, e, f; g, h, k; l, m, n; p, q, r, write 12, 1, 3; 1, 12, 3; 0, 14, 2; 4, 9, 3 respectively, and α will be $(= ep - dq = 1 \times 4 - 12 \times 9) = -104$, and β $(= dm - el = 12 \times 14 - 1 \times 0) = 168$, and so $\gamma = -143$, $\delta = 24$, $\zeta = -40$,

$= -40$, and $\theta = 33$. And therefore $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-3432 + 3432}{5720 - 5544}\right) = 0$; $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-104 + 0}{-143}\right) = \frac{8}{11}$, and $x \left(= \frac{p - gy - lz}{d} = \frac{4 - \frac{8}{11}}{12}\right) = \frac{3}{11}$. Wherefore, if there be mix'd $\frac{8}{11}$ Parts of a Pound of the second Mixture, $\frac{3}{11}$ Parts of a Pound of the third, and nothing of the first, the Aggregate will be a Pound, containing four Ounces of Silver, nine of Brass, and three of Tin.

PROBLEM IX. The Prices of several Mixtures of the same Things, and the Proportions of the Things mix'd together being given, to determine the Price of each of the Things mix'd.

Of each of the Things A, B, C, let the Price of the Mixture $dA + gB + lC$ be p, of the Mixture $eA + bB + mC$ the Price q, and of the Mixture $fA + kB + nC$ the Price r, and of those Things A, B, C let the Prices x, y, z be demanded. For the Things A, B, C substitute their Prices x, y, z, and there will arise the Æquations $dx + gy + lz = p$, $ex + by + mz = q$, and $fx + ky + nz = r$; from which, by proceeding as in the foregoing Problem, there will in like manner be got $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = z$, $\frac{\alpha + \beta z}{\gamma} = y$, and

$$\frac{p - gy - lz}{d} = x.$$

EXAMPLE. One bought 40 Bushels of Wheat, 24 Bushels of Barley, and 20 Bushels of Oats together, for 15 Pounds 12 Shillings. Again, he bought of the same Grain 26 Bushels of Wheat, 30 Bushels of Barley, and 50 Bushels of Oats together, for 16 Pounds: And thirdly, he bought of the like kind of Grain, 24 Bushels of Wheat, 120 Bushels of Barley, and 100 Bushels of Oats together, for 34 Pounds. It is demanded at what Rate a Bushel of each of the Grains ought to be valued. Answer, a Bushel of Wheat at 5 Shillings, of Barley at 3 Shillings, and of Oats at 2 Shillings. For instead of d, g, l; e, b, m; f, k, n; p, q, r, by writing respectively 40, 24, 20; 26, 30, 50; 24, 120, 100; $15\frac{3}{5}$, 16, and 34, there arises α $(= ep - dq = 26 \times 15\frac{3}{5} - 40 \times 16) = -234\frac{2}{5}$, and $\beta (= dm - el = 40 \times 50 - 26 \times 20) = 1480$, and thus $\gamma = -576$, $\delta = -500$, $\zeta = 1400$, and

and $\theta = -2400$. Then $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{562560 - 288000}{-806400 + 3552000} = \frac{274560}{2745600}\right) = \frac{1}{10}$; $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-234\frac{2}{5} + 148}{-576}\right) = \frac{3}{20}$; and $x \left(= \frac{p - gy - lz}{d} = \frac{15\frac{3}{5} - \frac{18}{5} - 2}{40}\right) = \frac{1}{4}$. Therefore a Buſhel of Wheat coſt $\frac{1}{4}$ ℔, or 5 Shillings, a Buſhel of Barley $\frac{3}{20}$ ℔, or 3 Shillings, and a Buſhel of Oats $\frac{1}{10}$ ℔, or 2 Shillings.

Problem X. There being given the ſpecifick Gravity both of the Mixture and the [two] Things mix'd, to find the Proportion of the mix'd Things to one another.

Let e be the ſpecifick Gravity of the Mixture $\overline{A + B}$, a the ſpecifick Gravity of A, and b the ſpecifick Gravity of B; and ſince the abſolute Gravity, or the Weight, is compoſed of the Bulk of the Body and the ſpecifick Gravity, aA will be the Weight of A, bB of B, and $eA + eB$ the Weight of the Mixture $\overline{A + B}$; and therefore $aA + bB = eA + eB$; and from thence $aA - eA = eB - bB$; and conſequently $e - b : a - e :: A : B$.

Example. Suppoſe the Gravity [or ſpecifick Weight] of Gold to be as 19, and of Silver as $10\frac{1}{3}$, and [King] *Hiero*'s Crown as 17; and $[6\frac{2}{3} : 2] :: 10 : 3$ $(e - b : a - e :: A : B)$ :: Bulk of Gold in the Crown : Bulk of Silver, or $190 : 31$ $(:: 19 \times 10 : 10\frac{1}{3} \times 3 :: a \times \overline{e - b} : b \times \overline{a - e})$ :: as the Weight of Gold in the Crown, to the Weight of Silver, and $221 : 31$ :: as the Weight of the Crown to the Weight of the Silver.

Problem XI. If the Number of Oxen a eat up the Meadow b in the Time c; and the Number of Oxen d eat up as good a Piece of Paſture e in the Time f, and the Graſs grows uniformly; to find how many Oxen will eat up the like Paſture g in the Time h.

If the Oxen a in the Time c eat up the Paſture b; then by Proportion, the Oxen $\frac{e}{b}a$ in the ſame Time c, or the Oxen $\frac{ec}{bf}a$ in the Time f, or the Oxen $\frac{ec}{bh}a$ in the Time h will

will eat up the Pasture e; supposing the Grass did not grow [at all] after the Time c. But since, by reason of the Growth of the Grass, all the Oxen d in the Time f can eat up only the Meadow e, therefore that Growth of the Grass in the Meadow e, in the Time $f-c$, will be so much as alone would be sufficient to feed the Oxen $d-\frac{eca}{bf}$ the Time f, that is as much as would suffice to feed the Oxen $\frac{df}{h}-\frac{eca}{bh}$ in the Time h. And in the Time $h-c$, by Proportion, so much would be the Growth of the Grass as would be sufficient to feed the Oxen $\frac{h-c}{f-c}$ into $\frac{df}{h}-\frac{eca}{bh}$ or $\frac{bdfh-ecah-bdcf+aecc}{bfh-bch}$. Add this Increment to the Oxen $\frac{aec}{bh}$, and there will come out $\frac{bdfh-ecah-bdcf+ecfa}{bfh-bch}$, the Number of Oxen which the Pasture e will suffice to feed in the Time h. And so by [in] Proportion the Meadow g will suffice to feed the Oxen $\frac{gbdfh-ecagh-bdcgf+ecfga}{befh-bceh}$ during the same Time h.

Example. If 12 Oxen eat up $3\frac{1}{3}$ Acres of Pasture in 4 Weeks, and 21 Oxen eat up 10 Acres of like Pasture in 9 Weeks; to find how many Oxen will eat up 36 Acres in 18 Weeks? Answer 36; for that Number will be found by substituting in $\frac{bdfgh-ecagh-bdcgf+ecfga}{befh-bceh}$ the Numbers 12, $3\frac{1}{3}$, 4, 21, 10, 9, 36, and 18 for the Letters a, b, c, d, e, f, g, and h respectively; but the Solution, perhaps, will be no less expedite, if it be brought out from the first Principles, in Form of the precedent literal Solution. As if 12 Oxen in 4 Weeks eat up $3\frac{1}{3}$ Acres, then by Proportion 36 Oxen in 4 Weeks, or 16 Oxen in 9 Weeks, or 8 Oxen in 18 Weeks, will eat up 10 Acres, on Supposition that the Grass did not grow. But since by reason of the Growth of the Grass 21 Oxen in 9 Weeks can eat up only 10 Acres, that Growth of the Grass in 10 Acres for the last 5 Weeks will be as much as would be sufficient to feed 5 Oxen,

Oxen, that is the Excefs of 21 above 16 for 9 Weeks, or, what is the fame Thing, to feed $\frac{5}{2}$ Oxen for 18 Weeks. And in 14 Weeks (the Excefs of 18 above the firft 4) the Increafe of the Grafs, by Analogy, will be fuch, as to be fufficient to feed 7 Oxen for 18 Weeks: Add thefe 7 Oxen, which the Growth of the Grafs alone would fuffice to feed, to the 8, which the Grafs without Growth after 4 Weeks would feed, and the Sum will be 15 Oxen. And, laftly, if 10 Acres fuffice to feed 15 Oxen for 18 Weeks, then, in Proportion, 24 Acres would fuffice 36 Oxen for the fame Time.

Problem XII. Having given the Magnitudes and Motions of Spherical Bodies perfectly elaftick, moving in the fame right Line, and meeting one another, to determine their Motions after Reflexion.

The Refolution of this Queftion depends on thefe Conditions, that each Body will fuffer as much by Re-action as the Action of each is upon the other, and that they muft recede from each other after Reflexion with the fame Velocity or Swiftnefs as they met before it. Thefe Things being fuppos'd, let the Velocity of the Bodies A and B, be a and b refpectively; and their Motions (as being compos'd of their Bulk and Velocity together) will be aA and bB. And if the Bodies tend the fame Way, and A moving more fwiftly follows B, make x the Decrement of the Motion aA, and the Increment of the Motion bB arifing by the Percuffion; and the Motions after Reflexion will be $aA-x$ and $bB+x$; and the Celerities $\frac{aA-x}{A}$ and $\frac{bB+x}{B}$, whofe Difference is $=$ to $a-b$ the Difference of the Celerities before Reflexion. Therefore there arifes this Æquation $\frac{bB+x}{B}-\frac{aA+x}{A}=a-b$, and thence by Reduction x becomes $=\frac{2aAB-2bAB}{A+B}$, which being fubftituted for x in the Celerities $\frac{aA-x}{A}$, and $\frac{bB+x}{B}$, there comes out $\frac{aA-aB+2bB}{A+B}$ for the Celerity of A, and $\frac{2aA-bA+bB}{A+B}$ for the Celerity of B after Reflexion.

But

But if the Bodies meet, then changing the Sign of b, the Velocities after Reflexion will be $\frac{aA-aB-2bB}{A+B}$, and $\frac{2aA+bA-bB}{A+b}$; either of which, if they come out, by Chance, Negative, it argues that Motion, after Reflexion, to tend a contrary Way to that which A tended to before Reflexion. Which is alſo to be underſtood of A's Motion in the former Caſe.

Example. If the homogeneous Bodies [or Bodies of the ſame Sort] A of 3 Pound with 8 Degrees of Velocity, and B a Body of 9 Pounds with 2 Degrees of Velocity, tend the ſame Way; then for A, a, B, and b, write 3, 8, 9, and 2; and $\left(\frac{aA-aB+2bB}{A+B}\right)$ becomes -1, and $\left(\frac{2aA-bA+bB}{A+B}\right)$ becomes 5. Therefore A will return back with one Degree of Velocity after Reflexion, and B will go on with 5 Degrees.

Problem XIII. To find 3 Numbers in continual Proportion, whoſe Sum ſhall be 20, and the Sum of their Squares 140?

Make the firſt of the Numbers $=x$, and the ſecond $=y$, and the third will be $\frac{yy}{x}$, and conſequently $x+y+\frac{yy}{x}=20$; and $xx+yy+\frac{y^4}{xx}=140$. And by Reduction $xx \begin{smallmatrix}+y\\-20\end{smallmatrix} x+yy=0$, and $x^4 \begin{smallmatrix}+yy\\-140\end{smallmatrix} xx+y^4=0$. Now to exterminate x, for a, b, c, d, e, f, g, h, in the third Rule, ſubſtitute reſpectively $1, 0, yy-140, 0, y^4$; $1, y-20$, and yy: and there will come out $\overline{-yy+280} \times y^6 : \overline{+2yy-40y} \times \overline{260y^4-40y^5} : +3y^4 \times y^4 : -2yy \times \overline{y^6-40y^5+400y^4} := 0$; and by Multiplication $1600y^6-10400y^5=0$, or $y=6\frac{1}{2}$. Which is found more ſhort by another Method before, but not ſo obvious as this. Moreover, to find x, ſubſtitute $6\frac{1}{2}$ for y in the Æquation $xx \begin{smallmatrix}+y\\-20\end{smallmatrix} x+yy=0$, and there will ariſe $xx-13\frac{1}{2}x$

$+42\frac{1}{4}$

$+42\frac{1}{4}=0$, or $xx=13\frac{1}{2}x-42\frac{1}{4}$, and having extracted the Root $x=6\frac{3}{4}+$, or $-\sqrt{3\frac{5}{16}}$; *viz.* $6\frac{3}{4}+\sqrt{3\frac{5}{16}}$ is the greatest of the three Numbers sought, and $6\frac{3}{4}$ and $\sqrt{3\frac{5}{16}}$ the least. For x denotes ambiguously either of the extreme Numbers, and thence there will come out two Values, either of which may be x, the other being $\frac{yy}{x}$.

The same otherwise. Putting the Numbers x, y, and $\frac{yy}{x}$ as before, you'll have $x+y+\frac{yy}{x}=20$, or $xx=\begin{matrix}20\\-y\end{matrix}x-yy$, and extracting the Root $x=10-\frac{1}{2}y+\sqrt{100-10y-\frac{3}{4}yy}$ for the first Number: Take away this and y from 20, and there remains $\frac{yy}{x}=10-\frac{1}{2}y-\sqrt{100-10y-\frac{3}{4}yy}$ the third Number. And the Sum of the Squares arising from these 3 Numbers is $400-40y$, and so $400-40y=140$, or $y=6\frac{1}{2}$. And having found the mean Number $6\frac{1}{2}$, substitute it for y in the first and third Number above found; and the first will become $6\frac{3}{4}+\sqrt{3\frac{5}{16}}$, and the third $6\frac{3}{4}-\sqrt{3\frac{5}{16}}$, as before.

Problem XIV. To find 4 Numbers in continual Proportion, the 2 Means whereof together make 12, and the 2 Extremes 20.

Let x be the second Number, and $12-x$ will be the third, $\frac{xx}{12-x}$ the first; and $\frac{144-24x+xx}{x}$ the fourth; and consequently $\frac{xx}{12-x}+\frac{144-24x+xx}{x}=20$. And by Reduction $xx=12x-30\frac{6}{7}$, or $x=6+\sqrt{5\frac{1}{7}}$. Which being found, the other Numbers are given from those above.

Problem XV. To find 4 Numbers continually proportional, whereof the Sum a is given, and [also] the Sum of their Squares b.

Altho' we ought for the most Part to seek the Quantities requir'd immediately, yet if there are 2 that are ambiguous, that is, that involve both the same Conditions, as here the 2 Means and 2 Extremes of the 4 Proportionals) the best Way is to seek other Quantities that are not ambiguous, by by which these may be determin'd, as suppose their Sum, or Difference, or Rectangle. Let us therefore make the Sum of the 2 mean Numbers to be s, and the Rectangle r, and the Sum of the Extremes will be $a-s$, and the Rectangle r, because of the Proportionality. Now that from hence these 4 Numbers may be found, make x the first, and y the second, and $s-y$ will be the third, and $a-s-x$ the fourth, and the Rectangle under the Means $sy-yy=r$, and thence one Mean $y=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-r}$, the other $s-y=\frac{1}{2}s-\sqrt{\frac{1}{4}ss-r}$. Also, the Rectangle under the Extremes $ax-sx-xx=r$, and thence one Extreme $x=\frac{a-s}{2}+\sqrt{\frac{ss-2as+aa}{4}-r}$, and the other $a-s-x=\frac{a-s}{2}-\sqrt{\frac{ss-2as+aa}{4}-r}$.

The Sum of the Squares of these 4 Numbers is $2ss-2as+aa-4r$, which is $=b$. Therefore $r=\frac{1}{2}ss-\frac{1}{2}as+\frac{1}{4}aa-\frac{1}{4}b$, which being substituted for r, there come out the 4 Numbers as follows:

The 2 Means $\begin{cases}\frac{1}{2}s+\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa} \\ \frac{1}{2}s-\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}.\end{cases}$

The 2 Extremes $\begin{cases}\frac{a-s}{2}+\sqrt{\frac{1}{4}b-\frac{1}{4}ss} \\ \frac{a-s}{2}-\sqrt{\frac{1}{4}b-\frac{1}{4}ss}\end{cases}$

Yet there remains the Value of s to be found. Wherefore, to abbreviate the Terms, for these Quantities substitute, and the 4 Proportionals will be

$$\frac{a-s}{2}+q$$
$$\frac{1}{2}s+p$$
$$\frac{1}{2}s-p$$
$$\frac{a-s}{2}-q$$

and make the Rectangle under the second and fourth equal to the Square of the third, since this Condition of the Question is not yet satisfy'd, and you'll have $\frac{as-ss}{4}-\frac{1}{2}qs+\frac{pa-ps}{2}-pq=\frac{1}{4}ss-ps+pp$. Make also the Rectangle

 under

under the first and third equal to the Square of the second, and you'll have $\frac{as-ss}{4}+\frac{1}{2}qs+\frac{-pa+ps}{2}-pq=\frac{1}{4}ss+ps+pp$. Take the first of these Æquations from the latter, and there will remain $qs-pa+ps=2ps$, or $qs=pa+ps$. Restore now $\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$ in the Place of p, and $\sqrt{\frac{1}{4}b-\frac{1}{4}ss}$ in the Place of q, and you'll have $s\sqrt{\frac{1}{4}b-\frac{1}{4}ss}=\overline{a+s}\sqrt{\frac{1}{4}b-\frac{1}{4}ss+\frac{1}{2}as-\frac{1}{4}aa}$, and by squaring $ss=-\frac{b}{a}s+\frac{1}{2}aa-\frac{1}{2}b$, or $s=-\frac{b}{2a}+\sqrt{\frac{bb}{4aa}+\frac{1}{2}aa-\frac{1}{2}b}$; which being found, the 4 Numbers sought are given from what has been shewn above.

PROBLEM XVI. If an annual Pension of the [Number of] Pounds a, to be paid in the five next following Years, be bought for the ready Money c, to find what the Compound Interest of 100 *l.* *per Annum* will amount to?

Make $1-x$ the Compound Interest of the Money x for a Year, that is, that the Money 1 to be paid after one Year is worth x in ready Money; and, by Proportion, the Money a to be paid after one Year will be worth ax in ready Money, and after 2 Years [it will be worth] axx, and after 3 Years ax^3, and after 4 Years ax^4, and after 5 Years ax^5. Add these 5 Terms, and you'll have $ax^5+ax^4+ax^3+axx+ax=c$, or $x^5+x^4+x^3+x^2+x=\frac{c}{a}$ an Æquation of 5 Dimensions; by Help of which when x is found by the Rules to be taught hereafter, put $x:1::100:y$, and $y-100$ will be the Compound Interest of 100 *l.* *per Annum.*

It is [will be] sufficient to have given these Instances in Questions where only the Proportions of Quantities are to be consider'd, without the Positions of Lines: Let us now proceed to the Solutions of Geometrical Problems.

How

How Geometrical Questions may be reduc'd to Æquations.

GEometrical Questions may be reduc'd sometimes to Æquations with as much Ease, and by the same Laws, as those we have propos'd concerning abstracted Quantities. As if the right Line AB be to be cut [or divided] in mean and extreme Proportion [or Reason] in C, that is, so that BE, the Square of the greatest Part, shall be equal to the Rectangle BD contain'd under the whole, and the least Part; having put $AB = a$, and $BC = x$, then will $AC = a - x$, and $xx = a$ into $a - x$; an Æquation which by Reduction gives $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}aa}$. [*Vide Figure* 6.]

But in Geometrical [Cases or] Affairs which more frequently occur, they so much depend on the various Positions and complex Relations of Lines, that they require some farther Invention and Artifice to bring them into Algebraick Terms. And tho' it is difficult to prescribe any Thing in these Sorts of Cases, and every Person's own Genius ought to be his Rule [or Guide] in these Operations; yet I'll endeavour to shew the Way to Learners. You are to know therefore, that Questions about the same Lines, related after any definite Manner to one another, may be variously propos'd, by making different Quantities the [*Quæsita*] or Things sought, from different [*Data*] or Things given. But of what *Data* or *Quæsita* soever the Question be propos'd, its Solution will follow the same Way by an Analytick Series, without any other Variation of Circumstance besides the feign'd Species of Lines, or the Names by which we are used to distinguish the given Quantities from those sought.

As if the Question be of an *Isosceles* CBD inscrib'd in a Circle, whose Sides BC, BD, and Base CD, are to be compar'd with the Diameter of the Circle AB. This may either be propos'd of the Investigation of the Diameter from the given Sides and Base, or of the Investigation of the Basis from the given Sides and Diameter; or lastly, of the Investigation of the Sides from the given Base and Diameter; but however it be propos'd, it will be reduc'd to an Æquation by the same Series of an Analysis. [*Vide Figure* 7.] *Viz.* If the Diameter be sought, I put $AB = x$, $CD = a$, and BC or $BD = b$. Then (having drawn AC) by reason

of

of the similar Triangles ABC, and CBE, AB will be to $BC :: BC : BE$, or $x : b :: b : BE$. Wherefore, $BE = \frac{bb}{x}$. Moreover, CE is $= \frac{1}{2}CD$, or to $\frac{1}{2}a$; and by reason of the right Angle CEB, $CEq + BEq, = BCq$, that is, $\frac{1}{4}aa + \frac{b^4}{xx} = bb$. Which Æquation, by Reduction, will give the Quantity x sought.

But if the Base be sought, put $AB = c$, $CD = x$, and BC or $BD = b$. Then (AC being drawn) because of the similar Triangles ABC and CBE, there is $AB : BC :: BC : BE$, or $c : b :: b : BE$. Wherefore $BE = \frac{bb}{c}$; and also $CE = \frac{1}{2}CD$, or $\frac{1}{2}x$. And because the Angle CEB is right, $CEq + BEq = BCq$, that is, $\frac{1}{4}xx + \frac{b^4}{cc} = bb$; an Æquation which will give by Reduction the sought Quantity x.

But if the Side BC or BD be sought, put $AB = c$, $CD = a$, and BC or $BD = x$. And (AC being drawn as before) by reason of the similar Triangles ABC and CBE, AB is to $BC :: BC : BE$, or $c : x :: x : BE$. Wherefore $BE = \frac{xx}{c}$. Moreover, CE is $= \frac{1}{2}CD$, or $\frac{1}{2}a$; and by reason of the right Angle CEB, $CEq + BEq = BCq$, that is, $\frac{1}{4}aa + \frac{x^4}{cc} = xx$; and the Æquation, by Reduction, will give the Quantity sought, *viz.* x.

You see therefore that in every Case, the Calculus, by which you come to the Æquation, is the same every where, and brings out the same Æquation, excepting only that I have denoted the Lines by different Letters according as I made the *Data* and *Quæsita* [different]. And from different *Data* and *Quæsita* there arises a Diversity in the Reduction of the Æquation found: For the Reduction of the Æquation $\frac{1}{4}aa + \frac{b^4}{xx} = bb$, in order to obtain $x = \frac{2bb}{\sqrt{4bb - aa}}$ the Value of AB, is different from the Reduction of the Æquation $\frac{1}{4}xx + \frac{b^4}{cc} = bb$, in order to obtain $x = \frac{2b}{c}\sqrt{cc - bb}$, the Value of CD; and the Reduction of the Æqua-

Æquation $\frac{1}{4}aa+\frac{x^4}{cc}=xx$ very different to obtain $x=\sqrt{\frac{1}{4}cc\pm\frac{1}{2}c\sqrt{cc-aa}}$ the Value of BC or BD: (as well as this also, $\frac{1}{4}aa+\frac{b^4}{cc}=bb$, to bring out c, a, or b, ought to be reduc'd after different Methods) but there was no Difference in the Investigation of these Æquations. And hence it is that [Analysts] order us to make no Difference between the given and sought Quantities. For since the same Computation agrees to any Case of the given and sought Quantities, it is convenient that they should be conceiv'd and compar'd without any Difference, that we may the more rightly judge of the Methods of computing them; or rather it is convenient that you should imagine, that the Question is propos'd of those [*Data* and *Quæsita*] given and sought Quantities, by which you think it is most easy for you to make out your Æquation.

Having therefore any Problem propos'd, compare the Quantities which it involves, and making no Difference between the given and sought ones, consider how they depend one upon another, that you may know what [Quantities] if they are assum'd, will, by proceeding Synthetically, give the rest. To do which, there is no need that you should at first of all consider how they may be deduc'd from one another Algebraically; but this general Consideration will suffice, that they may be some how or other deduc'd by a direct Connexion [with one another].

For Example, If the Question be put of the Diameter of the Circle AD, and the three Lines AB, BC, and CD inscrib'd in a Semi-circle, and from the rest given you are to find BC; at first Sight it is manifest, that the Diameter AD determines the Semi-circle, and then, that the Lines AB and CD by Inscription determine the Points B and C, and consequently the Quantity sought BC, and that by a direct Connexion; and yet after what Manner BC is to be had from these *Data* [or given Quantities] is not so evident to be found by an Analysis. The same Thing is also to be understood of AB or CD if they were to be sought from the other *Data*. [*Vide Figure* 8.] Now, if AD were to be found from the given Quantities AB, BC, and CD, it is equally evident it could not be done Synthetically; for the Distance of the Points A and D depends on the Angles B and

B and *C*, and thoſe Angles on the Circle in which the given Lines are to be inſcrib'd, and that Circle is not given without knowing the Diameter *AD*. The Nature of the Thing therefore requires, that *AD* be ſought, not Synthetically, but by aſſuming it [as given] to make thence a Regreſſion to the Quantities given.

When you ſhall have throughly perceiv'd the different Orderings [of the Proceſs] by which the Terms of the Queſtion may be explain'd, make Uſe of any of the Synthetical [Methods] by aſſuming Lines as given, from which you can form an eaſy Proceſs to others, tho' [the Regreſſion] to them may be very difficult. For the Computation, tho' it may proceed thro' various Mediums, yet will begin [originally] from thoſe [or ſuch] Lines; and will be ſooner perform'd by ſuppoſing the Queſtion to be ſuch, as if it was propos'd of thoſe *Data*, and ſome Quantity ſought that would eaſily come out from them, than by thinking of the Queſtion [in the Terms or Senſe] it is really propos'd in. Thus, in the propos'd Example, If from the reſt of the Quantities given you were to find *AD*: When I perceive that it cannot be done Synthetically, but yet that if it was done ſo, I could proceed in my Ratiocination on it in a direct Connexion [from one Thing] to others, I aſſume *AD* as given, and then I begin to compute as if it was given indeed, and ſome of the other Quantities, *viz.* ſome of the given ones, as *AB*, *BC*, or *CD*, were ſought. And by this Method, by carrying on the Computation from the Quantities aſſum'd after this Way to the others, as the Relations of the Lines [to one another] direct, there will always be obtain'd an Æquation between two Values of ſome one Quantity, whether one of thoſe Values be a Letter ſet down as a [Repreſentation or] Name at the Beginning of the Work for that Quantity, and the other a Value of it found out by Computation, or whether both be found by a Computation made after different Ways.

But when you have compar'd the Terms of the Queſtion thus generally, there is more Art and Invention requir'd to find out the particular Connexions or Relations of the Lines that ſhall accommodate them to [or render them fit for] Computation. For thoſe Things, which to a Perſon that does not ſo thoroughly conſider them, may ſeem to be immediately and by a very near Relation connected together, when we have a Mind to expreſs that Relation Algebraically, require a great deal more round-about Proceeding,

and

and oblige you to begin your Schemes anew, and carry on your Computation Step by Step; as may appear by finding *BC* from *AD*, *AB*, and *CD*. For you are only to proceed by such Propositions or Enunciations that can fitly be represented in Algebraick Terms, whereof in particular you have some from [*Eucl.*] *Ax.* 19. *Prop.* 4. *Book* 6. and *Prop.* 47. of the first.

In the first Place therefore, the Calculus may be assisted by the Addition and Subtraction of Lines, so that from the Values of the Parts you may find the Values of the whole, or from the Value of the whole and one of the Parts you may obtain the Value of the other Part.

In the second Place, the Calculus is promoted by the Proportionality of Lines; for we suppose (as above) that the Rectangle of the mean Terms, divided by either of the Extremes, gives the Value of the other; or, which is the same Thing, if the Values of all four of the Proportionals are first had, we make an Equality [or Æquation] between the Rectangles of the Extremes and Means. But the Proportionality of Lines is best found out by the Similarity of Triangles, which, as it is known by the Equality of their Angles, the Analyst ought in particular to be conversant in comparing them, and consequently not to be ignorant of *Eucl. Prop.* 5, 13, 15, 29, and 32 of the first Book, and of *Prop.* 4, 5, 6, 7, and 8 of the sixth Book, and of the 20, 21, 22, 27, and 31 of the third Book of his *Elem.* To which also may be added the 3d *Prop.* of the sixth Book, wherein, from the Proportion of the Sides is inferr'd the Equality of the Angles, and *e contra.* Sometimes likewise the 36 and 37th *Prop.* of the third Book will do the same Thing.

In the third Place, [the Calculus] is promoted by the Addition or Subtraction of Squares, *viz.* In right angled Triangles we add the Squares of the lesser Sides to obtain the Square of the greater, or from the Square of the greater Side we subtract the Square of one of the lesser, to obtain the Square of the other

And on these few Foundations (if we add to them *Prop.* 1. of the 6th *Elem.* when the Business relates to Superficies, as also some Propositions taken out of the 11th and 12th of *Euclid*, when Solids come in Question, the whole Analytick Art, as to right-lined Geometry, depends. Moreover, all the Difficulties of Problems may be reduc'd to the sole Composition of Lines out of Parts, and the Similarity of Triangles; so that there is no Occasion to make Use of other Theorems; because they may all be resolv'd into these two; and consequently into the Solutions that may be drawn from them. And, for an Instance of this, I have subjoin'd a Problem about letting fall a Perpendicular upon the Base of an oblique-angled Triangle, [which is] solv'd without the Help of the 47th *Prop.* of the first Book of *Eucl.* But altho' it may be of [great] Use not to be ignorant of the most simple Principles on which the Solutions of Problems depend, and tho' by only their Help any [Problems] may be solv'd; yet, for Expedition sake, it will be convenient not only that the 47th *Prop.* of the first Book of *Eucl.* whose Use is most frequent, but also that other Theorems should sometimes be made Use of.

As if [for Example] a Perpendicular being let fall upon the Base of an oblique angled Triangle, the Question were (for the sake of promoting Algebraick Calculus) to find the Segments of the Base; here it would be of Use to know, that the Difference of the Squares of the Sides is equal to the double Rectangle under the Base, and the Distance of the Perpendicular from the Middle of the Basis.

If the Vertical Angle of any Triangle be bisected, it will not only be of Use to know, that the Base may be divided in Proportion to the Sides, but also, that the Difference of the Rectangles made by the Sides, and the Segments of the Base is equal to the Square of the Line that bisects the Angle.

If the Problem relate to Figures inscrib'd in a Circle, this Theorem will frequently be of Use, *viz.* that in any quadrilateral Figure inscrib'd in a Circle, the Rectangle of the Diagonals is equal to the Sum of the Rectangles of the opposite Sides.

The Analyst may observe several Theorems of this Nature in his Practice, and reserve them for his Use; but let him use them sparingly, if he can, with equal Facility, or not much more Difficulty, hammer out the Solution from more simple Principles of Computation.

Wherefore let him take especial Notice of the three Principles first propos'd, as being more known, more simple, more general, but a few, and yet sufficient for all [Problems], and let him endeavour to reduce all Difficulties to them before others.

But that these Theorems may be accommodated to the Solution of Problems, the Schemes are oft times to be farther constructed, by producing out some of the Lines till they

they cut others, or become of an assign'd Length; or by drawing Lines parallel or perpendicular from some remarkable Point, or by conjoining some remarkable Points; as also sometimes by constructing after other Methods, according as the State of the Problem, and the Theorems which are made Use of to solve it, shall require. As for Example, If two Lines that do not meet each other, make given Angles with a certain third Line, perhaps we produce them so, that when they concur, or meet, they shall form a Triangle, whose Angles, and consequently the Reasons of their Sides, shall be given; or, if any Angle is given, or be equal to any one, we often complete it into a Triangle given in Specie, or similar to some other, and that by producing some of the Lines in the Scheme, or by drawing a Line subtending an Angle. If the Triangle be an oblique angled one, we often resolve it into two right angled ones, by letting fall a Perpendicular. If the Business concerns multilateral [or many sided] Figures, we resolve them into Triangles, by drawing Diagonal Lines; and so in others; always aiming at this End, *viz.* that the Scheme may be resolv'd either into given, or similar, or right angled Triangles. Thus, in the Example propos'd, I draw the Diagonal BD, that the Trapezium $ABCD$ may be resolv'd into the two Triangles, ABD a right angled one, and BDC an oblique angled one. [*Vide Figure 9.*] Then I resolve the oblique angled one into two right angled Triangles, by letting fall a Perpendicular from any of its Angles, B C or D, upon the opposite Side; as from B upon CD produc'd to E, that BE may meet it perpendicularly. But since the Angles BAD and BCD make in the mean while two right ones (by 22 *Prop.* 3 *Elem.*) as well as BCE and BCD, I perceive the Angles BAD and BCE to be equal; consequently the Triangles BCE and DAB to be similar. And so I see that the Computation (by assuming AD, AB, and BC as if CD were sought) may be thus carry'd on, *viz.* AD and AB (by reason of the right angled Triangle ABD) give you BD. AD, AB, BD, and BC (by reason of the similar Triangles ABD and CEB) give BE and CE. BD and BE (by reason of the right angled Triangle BED) give ED; and $ED - EC$ gives CD. Whence there will be obtain'd an Æquation between the Value of CD so found out, and the [small Algebraick] Letter that denotes it. We may also (and for the greatest Part it is better so to do, than to follow the Work too far in one continued Series) begin the

Computation from different Principles, or at least promote it by divers Methods to any one and the same Conclusion, that at length there may be obtain'd two Values of any the same Quantity, which may be made equal to one another. Thus, AD, AB, and BC, give BD, BE, and CE as before; then $CD + CE$ gives ED; and, lastly, BD, and ED (by reason of the right angled Triangle BED) give BE. You might also very well form the Computation thus, that the Values of those Quantities should be sought between which any other known Relation interceeds, and then that Relation will bring it to an Æquation. Thus, since the Relation between the Lines BD, DC, BC, and CE, is manifest from the 12th *Prop.* of the second Book of the *Elem. viz.* that $BDq - BCq - CDq = 2CD \times CE$: I seek BDq from the assum'd AD and AB; and CE from the assum'd AD, AB, and BC. And, lastly, assuming CD I make $BDq - BCq - CDq = 2CD \times CE$. After such Ways, and led by these Sorts of Consultations, you ought always to take care of the Series of the Analysis, and of the Scheme to be constructed in order to it, at once.

Hence, I believe, it will be manifest what Geometricians mean, when they bid you imagine that to be already done which is sought. For making no Difference between the known and unknown Quantities, you may assume any of them to begin your Computation from, as much as if all had [indeed] been known by a previous Solution, and you were no longer to consult the Solution of the Problem, but only the Proof of that Solution. Thus, in the first of the three Ways of computing already described, altho' perhaps AD be really sought, yet I imagine CD to be the Quantity sought, as if I had a mind to try whether its Value deriv'd from AD will coincide with [or be equal to] its Quantity before known. So also in the two last Methods, I don't propose, as my Aim, any Quantity to be sought, but only some how or other to bring out an Æquation from the Relations of the Lines: And, for sake of that Business I assume all [the Lines] AD, AB, BC, and CD as known, as much as if (the Question being before solv'd) the Business was to enquire whether such and such Lines would satisfy the Conditions of it, by [falling in or] agreeing with any Æquations which the Relations of the Lines can exhibit. I enter'd upon the Business at first Sight after this Way, and with such [Sort of] Consultations; but when I arrive at an Æquation, I change my Method, and endeavour

to

to find the Quantity sought by the Reduction and Solution of that Æquation. Thus, lastly, we assume often more Quantities as known, than what are expreſs'd in the State of the Question. Of this you may see an eminent Example in the 42d of the following Problems, where I have assum'd a, b, and c, in the Æquation $a + bx + cx^2 = yy$ for determining the Conick Section; as also the other Lines r, s, t, v, of which the Problem, as it is propos'd, hints nothing. For you may assume any Quantities by the Help whereof it is possible to come to Æquations; only taking this Care, that you obtain as many Æquations from them as you assume Quantities really unknown.

After you have consulted your Method of Computation, and drawn up your Scheme, give Names to the Quantities that enter into the Computation, (that is, from which being assum'd the Values of others are to be deriv'd, till at last you come to an Æquation) chusing such as involve all the Conditions of the Problem, and seem accommodated before others to the Business, and that shall render the Conclusion (as far as you can guess) more simple, but yet not more than what shall be sufficient for your Purpose. Wherefore, don't give proper Names to Quantities which may be denominated from Names already given. Thus, from a whole Line given and its Parts, from the three Sides of a right angled Triangle, and from three or four Proportionals, some one of the least considerable we leave without a Name, because its Value may be deriv'd from the Names of the rest. As in the Example already brought, if I make $AD = x$, and $AB = a$, I denote BD by no Letter, because it is the third Side of a right angled Triangle ABD, and consequently its Value is $\sqrt{xx - aa}$. Then if I say $BC = b$, since the Triangles DAB and BCE are similar, and thence the Lines $AD : AB :: BC : CE$ proportional, to three whereof, *viz.* to AD, AB, and BC there are already Names given; for that reason I leave the fourth CE without a Name, and in its room I make Use of $\frac{ab}{x}$ discover'd from the foregoing Proportionality. And so if DC be called c, I give no Name to DE, because from its Parts, DC and CE, or c and $\frac{ab}{x}$, its Value $c + \frac{ab}{x}$ comes out. [*Vide Figure* 10.]

But

But while I am talking of these Things, the Problem is almost reduc'd to an Æquation. For, after the aforesaid Letters are set down for the Species of the principal Lines, there remains nothing else to be done, but that out of those Species the Values of other Lines be made out according to a preconceiv'd Method, till after some foreseen Way they come to an Æquation. And I can see nothing wanting in this Case, except that by [means of] the right angled Triangles BCE and BDE I can bring out a double Value of BE, *viz.* $BCq - CEq$ $\left(\text{or } bb - \frac{aabb}{xx}\right) = BEq$; as also $BDq - DEq$ $\left(\text{or } xx - aa - cc - \frac{2abc}{x} - \frac{aabb}{xx}\right) = BEq$. And hence (blotting out on both Sides $\left.\frac{aabb}{xx}\right)$ I shall have the Æquation $bb = xx - aa - cc - \frac{2abc}{x}$; which being reduc'd, becomes $x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc$.

But since I have reckon'd up several Methods for the Solution of this Problem, and those not much unlike [one another] in the precedent [Paragraphs], of which that taken from *Prop.* 12. of the second Book of the *Elem.* being something cleverer than the rest, we will here subjoin it. Make therefore $AD = x$, $AB = a$, $BC = b$, and $CD = c$, and you'll have $BDq = xx - aa$, and $CE = \frac{ab}{x}$ as before. These Species therefore being substituted in the Theorem for $BDq - BCq - CDq = 2CD \times CE$, there will arise $xx - aa - bb - cc = \frac{2abc}{x}$, and after Reduction, $x^3 = \begin{matrix} aa \\ +bb \\ +cc \end{matrix} x + 2abc$, as before.

But that it may appear how great a Variety there is in the Invention of Solutions, and that it is not very difficult for a prudent Geometrician to light upon them, I have thought fit to teach [or shew] other Ways of doing the same Thing. And having drawn the Diagonal BD, if in room of the Perpendicular BE, which before was let fall from the Point B upon the Side DC, you now let fall a Perpendicular from the Point D upon the Side BC, or from the

the Point C upon the Side BD, by which the oblique angled Triangle BCD may any how be resolv'd into two right angled Triangles, you may come almost by the same Methods I have already describ'd to an Æquation. And there are other Methods very different from these; as if there are drawn two Diagonals, AC and BD, BD will be given by assuming AD and AB; as also AC by assuming AD and CD; then, by the known Theorem of Quadrilateral Figures inscrib'd in a Circle, *viz.* That $AD \times BC + AB \times CD$ is $= AC \times BD$, you'll obtain an Æquation. [*Vide Figure* 11.] Suppose therefore remaining the Names of the Lines AD, AB, BC, CD, *viz* x, a, b, c, BD will be $= \sqrt{xx - aa}$, and $AC = \sqrt{xx - cc}$, by the 47th *Prop.* of the first *Elem.* and these Species of the Lines being substituted in the Theorem we just now mention'd, there will come out $xb + ac = \sqrt{xx - cc} \times \sqrt{xx - aa}$. The Parts of which Æquation being squar'd and reduc'd, you'll again have $x^3 = \begin{matrix} aa \\ +bb \\ +cc \end{matrix} x + 2abc$.

But, moreover, that it may be manifest after what Manner the Solutions drawn from that Theorem may be thence reduc'd to only the Similarity of Triangles, erect BH perpendicular to BC, and meeting AC in H, and there will be form'd the Triangles BCH, BDA similar, by reason of the right Angles at B, and equal Angles at C and D, (by the 21. 3. *Elem.*) as also the Triangles BCD, BHA [which are also] similar, by reason of the equal Angles both at B, (as may appear by taking away the common Angle DBH from the two right ones) as also at D and A (by 21. 3 *Elem.*) You may see therefore, that from the Proportionality $BD : AD :: BC : HC$, there is given the Line HC; as also AH from the Proportionality $BD : CB :: AB : AH$. Whence since $AH + HC = AC$, you have an Æquation. The Names therefore aforesaid of the Lines remaining, *viz.* x, a, b, c, as also the Values of the Lines AC and BD, *viz.* $\sqrt{xx - cc}$ and $\sqrt{xx - aa}$, the first Proportionality will give $HC = \frac{xb}{\sqrt{xx - aa}}$, and the second will give $AH = \frac{ca}{\sqrt{xx - aa}}$. Whence, by reason of AH

$+ HC$

$+HC = AC$, you'll have $\frac{bx+ac}{\sqrt{xx-aa}} = \sqrt{xx-cc}$; an Æquation which (by multiplying by $\sqrt{xx-aa}$, and by squaring) will be reduc'd to a Form often describ'd in the preceding Pages.

But that it may yet farther appear what a Plenty of Solutions may be found, produce BC and AD till they meet in F, and the Triangles ABF and CDF will be similar, because the Angle at F is common, and the Angles ABF and CDF (while they compleat the Angle CDA to two right ones, by 13, 1. and 22, 3 *Elem.*) are equal. [*Vide Figure* 12.] Wherefore, if besides the four Terms which compose the Question, there was given AF, the Proportion $AB : AF :: CD : CF$ would give CF. Also $AF - AD$ would give DF, and the Proportion $CD : DF :: AB : BF$ would give BF; whence (since $BF - CF = BC$) there would arise an Æquation. But since there are assum'd two unknown Quantities as if they were given, there remains another Æquation to be found. I let fall therefore BG at right Angles upon AF, and the Proportion $AD : AB :: AB : AG$ will give AG; which being had, the Theorem in the 13, 2 *Eucl. viz.* that $BFq + 2FAG = ABq + AFq$ will give another Æquation. a, b, c, x remaining therefore as before, and making $AF = y$, you'll have (by insisting on the Steps already laid down) $\frac{cy}{a} = CF$. $y - x = DF$. $\frac{\overline{y-x} \times a}{c} = BF$. And thence $\frac{\overline{y-x} \times a}{c} - \frac{cy}{a} = b$, the first Æquation. Also $\frac{aa}{x}$ will be $= AG$, and consequently $\frac{aayy - 2a^2xy + a^2x^2}{cc} + \frac{2aay}{x} = aa + yy$ for the second Æquation. Which two, by Reduction, will give the Æquation sought, *viz.* The Value of y found by the first Æquation is $\frac{abc + aax}{aa - cc}$, which being substituted in the second, will give an Æquation, from which rightly order'd will come out $x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix}\, x + 2abc$, as before.

And

And so, if AB and DC are produc'd till they meet one another, the Solution will be much the same, unless perhaps it be something easier. Wherefore I will subjoyn another Specimen of this [Problem] from a Fountain very unlike the former, *viz.* by seeking the Area of the Quadrilateral Figure propos'd, and that doubly. I draw therefore the Diagonal BD, that the Quadrilateral Figure may be resolv'd into two Triangles. Then using the Names of the Lines x, a, b, c, as before, I find $BD = \sqrt{xx - aa}$, and $\frac{1}{2}a\sqrt{xx - aa}$ $(= \frac{1}{2}AB \times BD)$ the Area of the Triangle ABD. Moreover, having let fall BE perpendicularly upon CD you'll have (by reason of the similar Triangles ABD, BCE) $AD : BD :: BC : BE$, and consequently $BE = \frac{b}{x}\sqrt{xx - aa}$. Wherefore also $\frac{bc}{2x}\sqrt{xx - aa}$ $(= \frac{1}{2}CD \times BE)$ will be the Area of the Triangle BCD. Now, by adding these Area's, there will arise $\frac{ax + bc}{2x}\sqrt{xx - aa}$, the Area of the whole Quadrilateral. After the same Way, by drawing the Diagonal AC, and seeking the Area's of the Triangles ACD and ACB, and adding them, there will again be obtain'd the Area of the Quadrilateral Figure $\frac{cx + ba}{2x}\sqrt{xx - cc}$. Wherefore, by making these Area's equal, and multiplying both by $2x$, you'll have $\overline{ax + bc}\sqrt{xx - aa} = \overline{cx + ba}\sqrt{xx - cc}$, an Æquation which, by squaring and dividing by $aax - ccx$, will be reduc'd to the Form already often found out, $x^3 \begin{smallmatrix} -aa \\ +bb \\ +cc \end{smallmatrix} x + 2abc$.

Hence it may appear how great a Plenty of Solutions may be had, and that some Ways are much more neat than others. Wherefore, if the Method you take from your first Thoughts, for solving a Problem, be but ill accommodated to Computation, you must again consider the Relations of the Lines, till you shall have hit on a Way as fit and elegant as possible. For those Ways that offer themselves at first Sight, may create sufficient Trouble, perhaps, if they are made Use of. Thus, in the Problem we have been upon, nothing would have been more difficult than to have fallen upon the following Method instead of one of the precedent ones. [*Vide Figure* 13.] Having let fall BR and CS perpendicular to AD, as also CT to BR, the Figure will be resolv'd into right angled Triangles. And it may be seen, that AD and AB give AR, AD and CD give SD, $AD-AR-SD$ gives RS or TC. Also AB and AR give BR, CD and SD give CS or TR, and $BR-TR$ gives BT. Lastly, BT and TC give BC, whence an Æquation will be obtain'd. But if any one should go to compute after this Rate, he would fall into larger [and more perplex'd] Algebraick Terms than are any of the former, and more difficult to be brought to a final Æquation.

So much for the Solution of Problems in right lined Geometry; *unless it may* perhaps be worth while to note moreover, that when Angles, or Positions of Lines express'd by Angles, enter the State of the Question, Lines, or the Proportions of Lines ought to be used instead of Angles, *viz.* such as may be deriv'd from given Angles by a Trigonometrical Calculation; or from which being found, the Angles sought [will] come out by the same Calculus. Several Instances of which may be seen in the following Pages.

As for what belongs to the Geometry of Curve Lines, we use to denote them, either by describing them by the local Motion of right Lines, or by using Æquations indefinitely expressing the Relation of right Lines dispos'd [in order] according to some certain Law, and ending at the Curve Lines. The Antients did the same by the Sections of Solids, but less commodiously. But the Computations that regard Curves describ'd after the first Way, are no otherwise perform'd than in the precedent [Pages.] As if AKC be a Curve Line describ'd by K the Vertical Point of the Square $AK\varphi$, whereof one Leg AK freely slides through the Point A given by Position, while the other $K\varphi$ of a determinate Length is carry'd along the right Line AD also given by Position, and you are to find the Point C in which any right Line CD given [also] by Position shall cut this Curve: I draw the right Lines AC, CF, which may represent the Square in the Position sought, and the Relation of the Lines (without any Difference [or Regard] of what is given or sought, or any Respect had to the Curve) being consider'd, I perceive the Dependency of the others upon CF and any of these four, *viz.* BC, BF, AF, and AC to be Synthetical, two whereof I therefore assume, as $CF=a$, and

CB

$CB = x$, and beginning the Computation from thence, I presently obtain $BF = \sqrt{aa - xx}$, and $AB = \frac{xx}{\sqrt{aa - xx}}$, by reason of the right Angle CBF, and that the Lines $BF : BC :: BC : AB$ are continual Proportionals. Moreover, from the given Position of CD, AD is given, which I therefore call b, there is also given the reason of BC to BD, which I make as d to e, and you have $BD = \frac{ex}{d}$, and $AB = b - \frac{ex}{d}$. [*Vide Figure* 14.] Therefore $b - \frac{ex}{d} = \frac{xx}{\sqrt{aa - xx}}$, an Æquation which (by squaring its Parts and multiplying by $aa - xx$) will be reduc'd to this Form,

$$x^4 = \frac{2bdex^3 \begin{matrix} -bbdd \\ +aaee \end{matrix} xx - 2aabdex + aabbdd.}{dd + ee}$$

Whence, lastly, from the given Quantities a, b, d, and e, x may to be found by Rules hereafter to be given, and at that Interval [or Distance] x or BC, a right Line drawn parallel to AD will cut CD in the Point sought C.

Now, if we don't use Geometrical Descriptions but Æquations to denote the Curve Lines by, the Computations will thereby become as much shorter and easier, as the gaining of those Æquations can make them. As if the Intersection C of the given Ellipsis ACE with the right Line CD given by Position, be sought. To denote the Ellipsis, I take some known Æquation proper to it, as $rx - \frac{r}{q}xx = yy$, where x is indefinitely put for any Part of the Axis Ab or AB, and y for the Perpendicular bc or BC terminated at the Curve; and r and q are given from the given Species of the Ellipsis. [*Vide Figure* 15.]. Since therefore CD is given by Position, AD will be also given, which call a; and BD will be $a - x$; also the Angle ADC will be given, and thence the Reason of BD to BC, which call 1 to e, and BC (y) will be $= ea - ex$, whose Square $eeaa - 2eeax + eexx$ will be equal to $rx - \frac{r}{q}xx$. And thence

thence by Reduction there will arise $xx = \frac{2aeex + rx - aaee}{ee + \frac{r}{q}}$,

or $x = \frac{aee + \frac{1}{2}r \pm e\sqrt{ar + \frac{rr}{4ee} - \frac{aar}{q}}}{ee + \frac{r}{q}}$.

Moreover, altho' a Curve be denoted by a Geometrical Description, or by a Section of a Solid, yet thence an Æquation may be obtain'd, which shall define the Nature of the Curve, and consequently all the Difficulties of Problems propos'd about it may be reduc'd hither.

Thus, in the former Example, if AB be called x, and BC y, the third Proportional BF will be $\frac{yy}{x}$, whose Square, together with the Square of BC, is equal to CFq, that is, $\frac{y^4}{xx} + yy = aa$; or $y^4 + xxyy = aaxx$. And this is an Æquation by which every Point C of the Curve AKC, agreeing or corresponding to any Length of the Base (and consequently the Curve it self) is defin'd, and from whence consequently you may obtain the Solutions of Problems propos'd concerning this Curve.

After the same Manner almost, when a Curve is not given in Specie, but propos'd to be determin'd, you may feign an Æquation at Pleasure, that may generally contain its Nature, and assume this to denote it as if it was given, that from its Assumption you can any Way come to Æquations by which the Assumptions may be determin'd; Examples whereof you have in some of the following Problems, which I have collected for a more full Illustration of this Doctrine, and which I now proceed to deliver.

PRO-

PROBLEM I.

Having a finite right Line BC given, from whose Ends the two right Lines BA, CA are drawn with the given Angles ABC, ACB; to find AD the Height of the Concourse A, [or the Point of their Meeting] above the given Line BC. [Vide Figure 16.]

MAke $Bc = a$, and $AD = y$; and since the Angle ABD is given, there will be given (from the Table of Sines or Tangents) the Ratio between the Lines AD and BD which make as d to e. Therefore $d : e :: AD\ (y) : BD$. Wherefore $BD = \frac{ey}{d}$. In like Manner, by reason of the given Angle ACD there will be given the Ratio between AD and DC, which make as d to f, and you'll have $DC = \frac{fy}{d}$. But $BD + DC = BC$, that is, $\frac{ey}{d} + \frac{fy}{d} = a$. Which reduc'd by multiplying both Parts of the Æquation by d, and dividing by $e + f$, becomes $y = \frac{ad}{e + f}$.

PROBLEM II.

The Sides AB, AC of the Triangle ABC being given, and also the Base BC, which the Perpendicular AD [let fall] from the Vertical Angle cuts in D, to find the Segments BD and DC. [Vide Figure 17.]

LET $AB = a$, $AC = b$, $BC = c$, and $BD = x$, and DC will $= c - x$. Now since $ABq - BDq\ (aa - xx) = ADq$; and $ACq - DCq\ (bb - cc + 2cx - xx) = ADq$; you'll have $aa - xx = bb - cc + 2cx - xx$; which by Reduction becomes $\frac{aa - bb + cc}{2c} = x$.

But that it may appear that all the Difficulties of all Problems may be resolv'd by only the Proportionality of Lines, without the Help of the 47 of 1 *Eucl.* altho' not without round-about Methods, I thought fit to subjoyn the following Solution of this Problem over and above. From the Point D let fall the Perpendicular DE upon the Side AB, and the Names of the Lines, already given, remaining, you'll have $AB : BD :: BD : BE$.

$a : x :: x \frac{xx}{a}$. And $BA - BE \left(a - \frac{xx}{a}\right) = EA$, also $EA : AD :: AD : AB$, and consequently $EA \times AB\ (aa - xx) = ADq$. And so, by reasoning about the Triangle ACD, there will be found again $ADq = bb - cc + 2cx - xx$. Whence you will obtain as before $x = \frac{aa - bb + cc}{2c}$.

PROBLEM III.

The Area and Perimeter of the right angled Triangle ABC being given, to find the Hypothenuse BC. [Vide Figure 18.]

LET the Perimeter be [called] a, the Area bb, make $BC = x$, and $AC = y$; then will $AB = \sqrt{xx - yy}$; whence again the Perimeter $(BC + AC + AB)$ is $x + y + \sqrt{xx - yy}$, and the Area $(\frac{1}{2} AC \times AB)$ is $\frac{1}{2} y \sqrt{xx - yy} = bb$. Therefore $x + y + \sqrt{xx - yy} = a$, and $\frac{1}{2} y \sqrt{xx - yy} = bb$.

The latter of these Æquations gives $\sqrt{xx - yy} = \frac{2bb}{y}$; wherefore I write $\frac{2bb}{y}$ for $\sqrt{xx - yy}$ in the former Æquation, that the Asymmetry may be taken away; and there comes out $x + y + \frac{2bb}{y} = a$, or multiplying by y, and ordering [the Æquation] $yy = ay - xy - 2bb$. Moreover, from the Parts of the former Æquation I take away $x + y$, and there remains $\sqrt{xx - yy} = a - x - y$, and squaring the Parts to take away again the Asymmetry, there comes out $xx - yy = aa - 2ax - 2ay + xx + 2xy + yy$, which order'd and divided by 2, becomes $yy = ay - xy$

$+ aa$

$+ax - \frac{1}{2}aa$. Lastly, making an Equality between the 2 Values of yy, I have $ay - xy - 2bb = ay - xy + ax - \frac{1}{2}aa$, which reduc'd becomes $\frac{1}{2}a - \frac{2bb}{a} = x$.

The same otherwise.

Let $\frac{1}{2}$ the Perimeter $= a$, the Area $= bb$, and $BC = x$, and $AC + AB = 2a - x$. Now since xx (BCq) is $= ACq + ABq$, and $4bb = 2AC \times AB$, $xx + 4bb$ will $= ACq + ABq + 2AC \times AB =$ to the Square of $\overline{AC + AB}$ $=$ to the Square of $\overline{2a - x} = 4aa - 4ax + xx$. That is, $xx + 4bb = 4aa - 4ax + xx$, which reduc'd becomes $a - \frac{bb}{a} = x$.

Problem IV.

Having given the Area, Perimeter, and one of the Angles A of any Triangle ABC, to determine the rest. [Vide Figure 19.]

LET the Perimeter be $= a$, and the Area $= bb$, and from either of the unknown Angles, as C, let fall the Perpendicular CD to the opposite Side AB; and by reason of the given Angle A, AC will be to CD in a given Ratio, suppose as d to e. Call therefore $AC = x$, and CD will $= \frac{ex}{d}$, by which divide the double Area, and there will come out $\frac{2bbd}{ex} = AB$. Add AD (viz. $\sqrt{ACq - CDq}$, or $\frac{x}{d} \times \sqrt{dd - ee}$) and there will come out $BD = \frac{2bbd}{ex} + \frac{x}{d} \times \sqrt{dd - ee}$, to the Square whereof add CDq, and there will arise $\frac{4b^4dd}{eexx} + xx + \frac{4bb}{e}\sqrt{dd - ee} = BCq$. Moreover, from the Perimeter take away AC and AB, and there will remain $a - x - \frac{2bbd}{ex} = BC$, the Square whereof $aa - 2ax + xx - \frac{4abbd}{ex} + \frac{4bbd}{e} + \frac{4b^4dd}{eexx}$ make equal to the

the Square before found; and neglecting the Equivalents, you'll have $\frac{4bb}{e}\sqrt{dd-ee}=aa-2ax-\frac{4abbd}{ex}+\frac{4bbd}{e}$. And this, by assuming $4af$ for the given Terms $aa+\frac{4bbd}{e}-\frac{4bb}{e}\sqrt{dd-ee}$; and by reducing becomes $xx=2fx-\frac{2bbd}{e}$, or $x=f\pm\sqrt{ff-\frac{2bbd}{e}}$.

The same Æquation would have come out also by seeking the Leg AB; for the Sides AB and AC are indifferently alike to all the Conditions of the Problem. Wherefore if AC be made $=f-\sqrt{ff-\frac{2bbd}{e}}$, AB will $=f+\sqrt{ff-\frac{2bbd}{e}}$, and reciprocally; and the Sum of these $2f$ subtracted from the Perimeter, leaves the third Side $BC=a-2f$.

PROBLEM V.

Having given the Altitude, Base, and Sum of the Sides, to find the Triangle.

LET the Altitude $CD=a$, half the Basis $AB=b$, half the Sum of the Sides $=c$, and their Semi-difference $=z$; and the greater Side as $BC=c+z$, and the lesser $AC=c-z$. Subtract CDq from CBq, and also from ACq, and hence will $BD=\sqrt{cc+2cz+zz-aa}$, and thence $AD=\sqrt{cc-2cz+zz-aa}$. Subtract also AB from BD, and AD will again $=\sqrt{cc+2cz+zz-aa}-2b$. Having now squared the Values of AD, and order'd the Terms, there will arise $bb+cz=b\sqrt{cc+2cz+zz-aa}$. Again, by squaring and reducing into Order, you'll obtain $cczz-bbzz=bbcc-bbaa-b^4$. And $z=b\sqrt{1-\frac{aa}{cc-bb}}$. Whence the Sides are given.

PRO-

PROBLEM VI.

Having given the Base AB, and the Sum of the Sides $AC + BC$, and also the Vertical Angle C, to determine the Sides. [Vide Figure 20.]

MAke the Base $= a$, half the Sum of the Sides $= b$, and half the Difference $= x$, and the greater Side BC will be $= b + x$, and the lesser $AC = b - x$. From either of the unknown Angles A let fall the Perpendicular AD to the opposite Side BC, and by reason of the given Angle C there will be given the Ratio of AC to CD, suppose as d to e, and then CD will $= \frac{eb - ex}{d}$. Also, by 11. 2 *Elem.* $\frac{ACq - ABq + BCq}{2BC}$, that is, $\frac{2bb + 2xx - aa}{2b + 2x} = CD$; and so you have an Æquation between the Values of CD. And this reduc'd, x becomes $= \sqrt{\frac{daa + 2ebb - 2dbb}{2d + 2e}}$, whence the Sides are given.

If the Angles at the Base were sought, the Conclusion would be more neat, as draw EC bisecting the given Angle and meeting the Base in E; and $AB : AC + BC$ (:: $AE : AC$) :: Sine Angle ACE : Sine Angle AEC; and if from the Angle AEC, and also from its Complement BEC you subtract $\frac{1}{2}$ the Angle C, there will be left the Angles ABC and BAC.

PROBLEM VII.

Having given the Sides of any Parallelogram AB, BD, DC, and AC, and one of the Diagonals BC, to find the other Diagonal AD. [Vide Figure 21.]

LET E be the Concourse of the Diagonals, and to the Diagonal BC let fall the Perpendicular AF, and by the 13. 2 *Elem.* $\frac{ACq - ABq + BCq}{2BC} = CF$. And also

ACq

$\frac{ACq - AEq + ECq}{2EC} = CF$. Wherefore since $EC = \frac{1}{2}BC$, and $AE = \frac{1}{2}AD$, $\frac{ACq - ABq + BCq}{2BC} = \frac{ACq - \frac{1}{4}ADq + \frac{1}{4}BCq}{BC}$, and having reduc'd, $AD = \sqrt{2ACq + 2ABq - BCq}$.

Whence, by the by, in any Parallelogram, the Sum of the Squares of the Sides is equal to the Sum of the Squares of the Diagonals.

PROBLEM VIII.

Having given the Angles of the Trapezium ABCD, also its Perimeter and Area, to determine the Sides. [Vide Figure 22.]

PRoduce any two of the Sides AB and DC till they meet in E, and let $AB = x$, and $BC = y$, and because all the Angles are given, there are given the Ratio's of BC to CE and BE, which make d to e and f; and CE will be $= \frac{ey}{d}$, and $BE = \frac{fy}{d}$, and consequently $AE = x + \frac{fy}{d}$. There are also given the Ratio's of AE to AD, and of AE to DE; which make as g to d, and as h to d; and AD will $= \frac{dx + fy}{g}$, and $ED = \frac{dx + fy}{h}$, and consequently $CD = \frac{dx + fy}{h} - \frac{ey}{d}$, and the Sum of all the Sides $x + y + \frac{dx + fy}{g} + \frac{dx + fy}{h} - \frac{ey}{d}$; which, since it is given, call it a, and the Terms will be abbreviated by writing $\frac{p}{r}$ for the given $1 + \frac{d}{g} + \frac{d}{h}$, and $\frac{q}{r}$ for the given $1 + \frac{f}{g} + \frac{f}{h} - \frac{e}{d}$, and you'll have the Æquation $\frac{px + qy}{r} = a$.

Moreover, by Reason of all the Angles given, there is given the [Ratio or] Reason of BCq to the Triangle BCE, which make as m to n, and the Triangle $BCE = \frac{n}{m}yy$. There is also given the Ratio of AEq to the Triangle ADE;

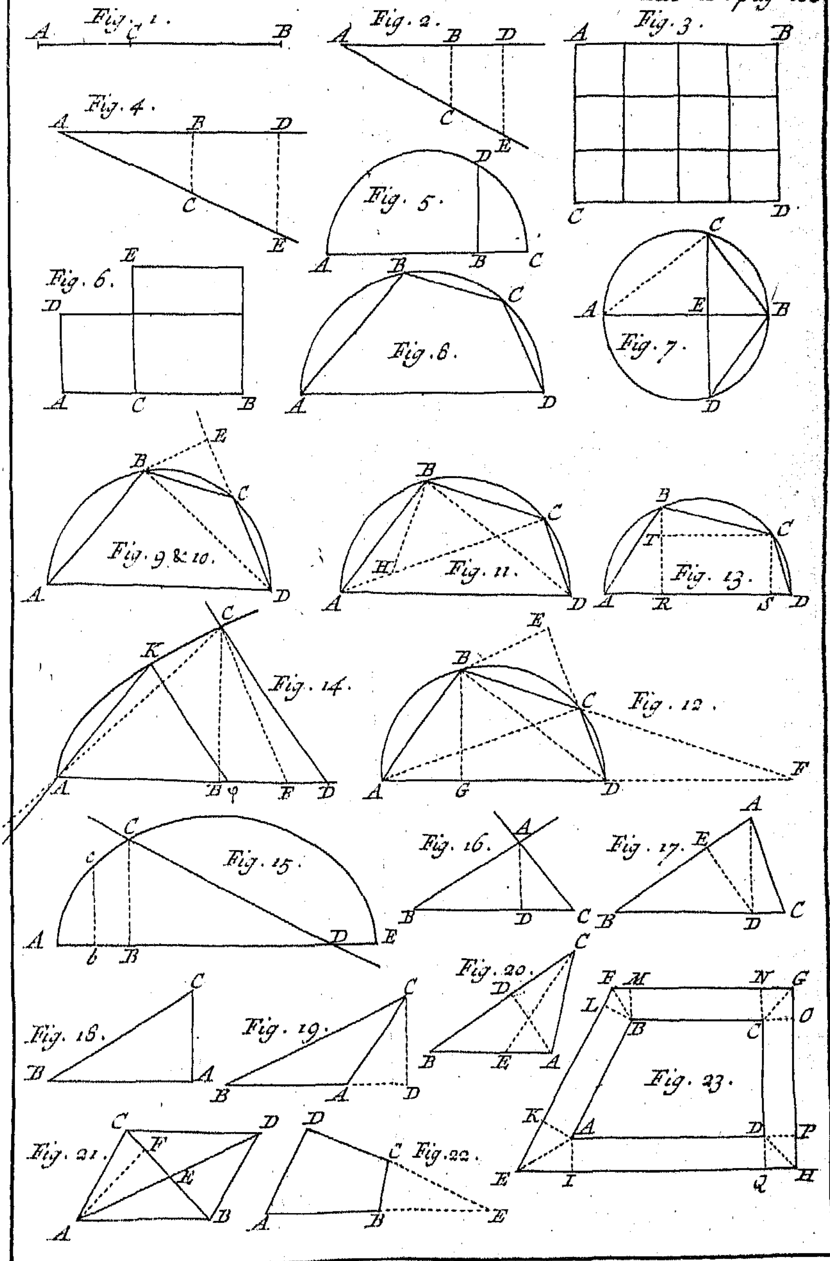


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 8.
Fig. 7.
Fig. 9 & 10.
Fig. 11.
Fig. 13.
Fig. 14.
Fig. 12.
Fig. 15.
Fig. 16.
Fig. 17.
Fig. 20.
Fig. 18.
Fig. 19.
Fig. 23.
Fig. 21.
Fig. 22.

ADE; which make as m to d; and the Triangle *ADE* will be $= \frac{ddxx + 2dfxy + ffyy}{dm}$. Wherefore, since the Area *AC*, which is the Difference of those Triangles, is given, let it be bb, and $\frac{ddxx + 2dfxy + ffyy - dnyy}{dm}$ will be $= bb$. And so you have two Æquations, from [or by] the Reduction whereof all is determin'd. *Viz.* The Æquation above gives $\frac{ra - qy}{p} = x$, and by writing $\frac{ra - qy}{p}$ for x in the last, there comes out $\frac{drraa - 2dqray + dqqyy}{ppm} + \frac{2afry - 2fqyy}{pm} + \frac{ffyy - dnyy}{dm} = bb$; and the Terms being abbreviated by writing s for the given Quantity $\frac{dqq}{pp} - \frac{2fq}{p} + \frac{ff}{d} - n$, and $-st$ for the given $-\frac{adqr}{pp} + \frac{afr}{p}$, and stv for the given $bbm - \frac{drraa}{pp}$, there arises $yy = 2ty + tv$, or $y = t + \sqrt{tt + tv}$.

Problem IX.

To surround the Fish-Pond ABCD with a Walk ABCDEFGH of a given Area, and of the same Breadth every where. [Vide Figure 23.]

LET the Breadth of the Walk be x, and its Area aa. And, letting fall the Perpendiculars *AK*, *BL*, *BM*, *CN*, *CO*, *DP*, *DQ*, *AI*, from the Points *A*, *B*, *C*, *D*, to the Lines *EF*, *FG*, *GH*, and *HE*, to divide the Walk into the four Trapezia *IK*, *LM*, *NO*, *PQ*, and into the four Parallelograms *AL*, *BN*, *CP*, *DI*, of the Latitude x, and of the same Length with the Sides of the given Trapezium. Let therefore the Sum of the Sides $(AB + BC + CD + DA) = b$, and the Sum of the Parallelograms will be $= bx$.

Moreover, having drawn AE, BF, CG, DH; since AI is $= AK$, the Angle AEI will be $=$ Angle AEK $= \frac{1}{2} IEK$, or $\frac{1}{2} DAB$. Therefore the Angle AEI is given, and consequently the Ratio of AI to IE, which make as d to e, and IE will be $= \frac{ex}{d}$. Multiply this into $\frac{1}{2} AI$, or $\frac{1}{2} x$, and the Area of the Triangle AEI will be $= \frac{exx}{2d}$. But by reason of equal Angles and Sides, the Triangles AEI and AEK are equal, and consequently the Trapezium IK ($=$ 2 Triang. AEI) $= \frac{exx}{d}$. In like manner, by putting $BL : LF :: d : f$, and $CN : NG :: d : g$, and $DP : DH :: d : h$, (for those Reasons are also given from the given Angles B, C, and D) you'll have the Trapezium $LM = \frac{fxx}{d}$, $NO = \frac{gxx}{d}$, and $PQ = \frac{hxx}{d}$. Wherefore $\frac{exx}{d} + \frac{fxx}{d} + \frac{gxx}{d} + \frac{hxx}{d}$, or $\frac{pxx}{d}$, by writing p for $e + f + g + h$ will be equal to the four Trapeziums $IK + LM + NO + PQ$; and consequently $\frac{pxx}{d} + bx$ will be equal to the whole Walk aa. Which Æquation, by dividing all the Terms by $\frac{p}{d}$, and extracting its Root, x will become $= \frac{-db + \sqrt{bbdd + 4aapd}}{2p}$. And the Breadth of the Walk being thus found, it is easy to describe it.

Problem X.

From the given Point C, to draw the right Line CF, which [together] with two other right Lines AE and AF given by Position, shall comprehend [or constitute] the Triangle AEF of a given Magnitude. [Vide Figure 24.]

DRAW CD parallel to AE, and CB and EG perpendicular to AF, and let $AD = a$, $CB = b$, and $AF = x$, and the Area of the Triangle $AEF = cc$, and by reason

reason of the proportional Quantities $DF:AF(::DC:AE)::CB:EG$, that is, $a+x:x::b:\frac{bx}{a+x}$ will be $\frac{bx}{a+x}=EG$. Multiply this into $\frac{1}{2}AF$, and there will come out $\frac{bxx}{2a+2x}$, the Quantity of the Area AEF, which is $=cc$. And so the Æquation being order'd [rightly] xx will $=\frac{2ccx+2cca}{b}$, or $x=\frac{cc+\sqrt{c^4+2ccab}}{b}$.

After the same Manner a right Line may be drawn thro' a given Point, which shall divide any Triangle or Trapezium in a given Ratio.

Problem XI.

To determine the Point C in the given right Line DF, from which the right Lines AC and BC drawn to two other Points A and B given by Position, shall have a given Difference. [Vide Figure 25.]

FROM the given Points let fall the Perpendiculars AD and BF to the given right Line, and make $AD=a$, $BF=b$, $DF=c$, $DC=x$, and AC will $=\sqrt{aa+xx}$, $FC=x-c$, and $BC=\sqrt{bb+xx-2cx+cc}$. Now let their given Difference be d, AC being greater than $BC\times$ $\sqrt{aa+xx}-d$ will $=\sqrt{bb+xx-2cx+cc}$. And squaring the Parts $aa+xx+dd-2d\sqrt{aa+xx}=bb+xx-2cx+cc$. And reducing, and for Abbreviation sake, writing $2ee$ instead of the given [Quantities] $aa+dd-bb-cc$, there will come out $ee+cx=d\times\sqrt{aa+xx}$. And again, having squared the Parts $e^4+2ceex+ccxx=ddaa+ddxx$. And the Æquation being reduc'd $xx=\frac{2eecx+e^4-aadd}{dd-cc}$, or

$$x=\frac{eec+\sqrt{e^4dd-aad^4+aaddcc}}{dd-cc}.$$

The

The Problem will be resolv'd after the same Way, if the Sum of the Lines AC and BC, or the Sum of the Difference of their Squares, or the Proportion or Rectangle, or the Angle comprehended by them be given: Or also, if instead of the right Line DC, you make Use of the Circumference of a Circle, or any other Curve Line, so the Calculation (in this last Case especially) relates to the Line that joyns the Points A and B.

PROBLEM XII.

To determine the Point Z, from which if four right Lines ZA, ZB, ZC, and ZD are drawn at given Angles to four right Lines given by Position, viz. *FA, EB, FC, GD, the Rectangle of two of the given Lines ZA and ZB, and the Sum of the other two ZC and ZB may be given.* [Vide Figure 26.]

FRom among the Lines chuse one, as FA, given by Position, as also another, ZA, not given by Position, and which is drawn to it, from the Lengths whereof the Point Z may be determin'd, and produce the other Lines given by Position till they meet these, or be produc'd farther out if there be Occasion, as you see [here]. And having made $EA = x$, and $AZ = y$, by reason of the given Angles of the Triangles AEH, there will be given the Ratio of AE to AH, which make as p to q, and AH will be $= \frac{qx}{p}$. Add AZ, and ZH will be $= y + \frac{qx}{p}$. And thence, since by reason of the given Angles of the Triangle HZB, there is given the Ratio of HZ to BZ, if that be made as n to p you'll have $ZB = \frac{py + qx}{n}$. Moreover, if the given EF be called a, AF will $= a - x$, and thence, if by reason of the given Angles of the Triangle AFI, AF be made to AI in the same Ratio as p to r, AI will become $= \frac{ra - rx}{p}$. Take this from AZ and there will remain

$IZ = y - \frac{ra - rx}{p}$. And by reason of the given Angles of the Triangle ICZ, if you make IZ to ZC in the same Ratio as m to p, ZC will become $= \frac{py - ra + rx}{m}$. After the same Way, if you make $EG = b$. $AG : AK :: l : s$, and $ZK : ZD :: p : l$, there will be obtain'd $ZD = \frac{sb - sx - ly}{p}$.

Now, from the State of the Question, if the Sum of the two [Lines] ZC and ZD, $\frac{py - ra + rx}{m} + \frac{sb - sx - ly}{p}$ be made equal to any given one; and the Rectangle of the other two $\frac{pyy + qxy}{n}$ be made $= gg$, you'll have two Æquations for determining x and y. By the latter there comes out $x = \frac{ngg - pyy}{qy}$, and by writing this Value of x in the room of that in the former Æquation, there will come out $\frac{py - ra}{m} + \frac{rngg - rpyy}{mqy} + \frac{sb - ly}{p} - \frac{sngg - spyy}{pqy} = f$; and by Reduction $yy = \frac{apqry - bmqsy + fmpqy + ggmns - ggnpr}{ppq - ppr - mpq + mps}$; and for Abbreviation sake, writing $2h$ for $\frac{apqr - bmqs + fmpq}{ppq - ppr - mlq + mps}$, and kk for $\frac{ggmns - ggpnr}{ppq - ppr - mlq + mps}$, you'll have $yy = 2hy + kk$, or $y = h \pm \sqrt{hh + kk}$. And since y is known by means of this Æquation, the Æquation $\frac{ngg - pyy}{qy} = x$ will give x. Which is sufficient to determine the Point z.

After the same Way a Point is determin'd [or may be determin'd] from which other right Lines may be drawn to more or fewer right Lines given by Position, so that the Sum, or Difference, or Rectangle of some of them may be given, or may be made equal to the Sum, or Difference, or Rectangle of the rest, or that they may have any other assign'd Conditions.

PRO-

PROBLEM XIII.

To subtend the right Angle EAF with the right Line EF given in Magnitude, which shall pass through the given Point C, [which shall be] æquidistant from the Lines that comprehend the right Angle (when they are produc'd). [Vide Figure 27.]

COmplete the Square $ABCD$, and bisect the Line EF in G. Then call CB or CD, a; EG or FG, b; and CG, x; and CE will $= x - b$, and $CF = x + b$. Then since $CFq - BCq = BFq$, BF will $= \sqrt{xx + 2bx + bb - aa}$. Lastly, by reason of the similar Triangles CDE, FBC, $CE : CD :: CF : BF$, or $x - b : a :: x + b : \sqrt{xx + 2bx + bb - aa}$. Whence $ax + ab = \overline{x - b} \times \sqrt{xx + 2bx + bb - aa}$. Each part of which Æquation being squar'd, and the Terms that come out being reduc'd into Order, there comes out $x^4 = \begin{matrix} 2aa \\ + 2bb \end{matrix} xx \begin{matrix} + 2aabb \\ - b^4 \end{matrix}$. And extracting the Root as in Quadratick Æquations, there comes out $xx = aa + bb + \sqrt{a^4 + 4aabb}$; and consequently $x = \sqrt{aa + bb + \sqrt{a^4 + 4aabb}}$. And CG being thus found, gives CE or CF, which, by determining the Point E or F, satisfies the Problem.

The same otherwise:

Let CE be $= x$, $CD = a$, and $EF = b$; and CF will be $= x + b$, and $BF = \sqrt{xx + bb + 2bx - aa}$. And then since $CE : CD :: CF : BF$, or $x : a :: x + b : \sqrt{xx + 2bx + bb - aa}$, $ax + ab$ will be $= x \times \sqrt{xx + 2bx + bb - aa}$. The Parts of this Æquation being squar'd, and the Terms reduc'd into Order, there will come out $x^4 + 2bx^3 \begin{matrix} + bb \\ - 2aa \end{matrix} xx - 2aabx - aabb = 0$, a Biquadratick Æquation, the Investigation of the Root of which is more difficult than in the former Case. But it may

may be thus investigated; put $x^4 + 2bx^3 \begin{matrix} +bb \\ -2aa \end{matrix} xx = 2aabx + a^4 = aabb + a^4$, and extracting the Root on both Sides $xx + bx - aa = \pm a\sqrt{aa+bb}$.

Hence I have an Opportunity of giving a Rule for the Election of Terms for the Calculus, *viz.* when there happens to be such an Affinity or Similitude of the Relation of two Terms to the other Terms of the Question, that you should be oblig'd in making Use of either of them to bring out Æquations exactly alike; or that both, if they are made Use of together, shall bring out the same Dimensions and the same Form in the final Æquation, (only excepting perhaps the Signs + and —, which will be easily [and readily] seen) then it will be the best Way to make Use of neither of them, but in their room to chuse some third, which shall bear a like Relation to both, as suppose the half Sum, or half Difference, or perhaps a mean Proportional, or any other Quantity related to both indifferently and without a like [before made Use of]. Thus, in the precedent Problem, when I see the Line EF alike related to both AB and AD, (which will be evident if you also draw EF in the Angle BAH) and therefore I can by no Reason be perswaded why ED should be rather made Use of than BF, or AE rather than AF, or CE rather than CF for the Quantity sought: Wherefore, in the room of the Points C and F, from whence this Ambiguity comes, (in the former Solution) I made Use of the intermediate [Point] G, which has [or bears] a like Relation to both the Lines AB and AD. Then from this Point G, I did not let fall a Perpendicular to AF for finding the Quantity sought, because I might by the same Ratio have let one fall to AD. And therefore I let it fall upon neither CB nor CD, but propos'd CG for the Quantity sought, which does not admit of a like; and so I obtain'd a Biquadratick Æquation without the odd Terms.

I might also (taking Notice that the Point G lies in the Periphery of a Circle describ'd from the Center A, by the Radius EG) have let fall the Perpendicular GK upon the Diagonal AC, and have sought AK or CK, (as which bear also a like Relation to both AB and AD) and so I should have fall'n upon a Quadratick Æquation, *viz.* $yy = \frac{1}{2}ey + \frac{1}{2}bb$, making $AK = y$, $AC = e$, and $AG = b$. And AK being so found, there must have been erected the Perpendicular

 pendicular

pendicular KG meeting the aforesaid Circle in G, thro' which CF would pass.

Taking particular Notice of this Rule in *Prob.* 5 *and* 6, where the Sides BC and AC were to be determin'd, I rather sought the Semi-difference than either of them. But the Usefulness of this Rule will be more evident from the following Problem.

PROBLEM XIV.

So to inscribe the right Line DC of a given Length in the given Conick Section DAC, that it may pass through the Point G given by Position. [Vide Figure 28.]

LET AF be the Axis of the Curve, and from the Points D, G, and C let fall to it the Perpendiculars DH, GE, and CB. Now to determine the Position of the right Line DC, it may be propos'd to find out the Point D or C; but since these are related, and so alike, that there would be the like Operation in determining either of them, whether I were to seek CG, CB, or AB; or their likes, DG, DH, or AH; therefore I look after a third Point, that regards D and C alike, and at the same time determines them. And I see F to be such a Point.

Now let $AE=a$, $EG=b$, $DC=c$, and $EF=z$; and besides, since the Relation between AB and BC in the Æquation, I suppose, given for determining the Conick Section, let $AB=x$, $BC=y$, and FB will be $=x-a+z$. And because $GE:EF::CB:FB$, FB will again be $=\frac{yz}{b}$. Therefore, $x-a+z=\frac{yz}{b}$. These Things being thus laid down, take away x, by the Equation that denotes [or expresses] the Curve. As if the Curve be a Parabola express'd by the Æquation $rx=yy$, write $\frac{yy}{r}$ for x; and there will arise $\frac{yy}{r}-a+z=\frac{yz}{b}$, and extracting the Root $y=\frac{rz}{2b}\pm\sqrt{\frac{rrzz}{4bb}+ar-rz}$. Whence it is evident, that

that $\sqrt{\frac{rrzz}{bb} + 4ar - 4rz}$ is the Difference of the double Value of y, that is, of the Lines $+BC$ and $-DH$, and consequently (having let fall DK perpendicular upon CB) that Difference is equal to CK. But $FG : GE :: DC : CK$, that is, $\sqrt{bb+zz} : b :: c : \sqrt{\frac{rrzz}{bb} + 4ar - 4rz}$. And by multiplying the Squares of the Means, and also the Squares of the Extreams into one another, and ordering the Products, there will arise $z^4 =$

$$\frac{4bbrz^3 \begin{matrix} -4abbr \\ -bbrr \end{matrix} zz + 4b^4rz \begin{matrix} -4ab^4r \\ +b^4cc \end{matrix}}{rr},$$ an Æquation of four Dimensions, which would have risen to one of eight Dimensions if I had sought either CG, or CB, or AB.

PROBLEM XV.

To multiply or divide a given Angle by a given Number. [Vide Figure 29.]

IN any Angle FAG inscribe the Lines AB, BC, CD, DE, &c. of any the same Length, and the Triangles ABC, BCD, CDE, DEF, &c. will be *Isosceles*, and consequently by the 32. 1. *Eucl.* the Angle CBD will be $=$ Angle $A + ACB = 2$ Angle A, and the Angle $DCE =$ Angle $A + ADC = 3$ Angle A, and the Angle $EDF =$ $A + AED = 4$ Angle A, and the Angle $FEG = 5$ Angle A, and so onwards. Now, making AB, BC, CD, &c. the Radii of equal Circles, the Perpendiculars BK, CL, DM, &c. let fall upon AC, BD, CE, &c. will be the Sines of those Angles, and AK, BL, CM, DN, &c. will be their Sines Complement to a right one; or making AB the Diameter, the Lines AK, BL, CM, &c. will be Chords. Let therefore $AB = 2r$, and $AK = x$, then work thus:

$$AB : AK :: AC : AL.$$
$$2r : x :: 2x : \frac{xx}{r}.$$

And

And $\left\{\begin{matrix} AL - AB \\ \frac{xx}{r} - 2r \end{matrix}\right\} = BL$, the Duplication.

$$AB : AK :: AD\ (2AL - AB) : AM.$$

$$2r : x :: \frac{2xx}{r} - 2r : \frac{x^3}{rr} - x.$$

And $\left\{\begin{matrix} AM - AC \\ \frac{x^3}{rr} - 3x \end{matrix}\right\} = CM$, the Triplication.

$$AB : AK :: AE\ (2AM - AC) : AN.$$

$$2r : x :: \frac{2x^3}{rr} - 4x : \frac{x^4}{r^3} - \frac{2xx}{r}.$$

And $\left\{\begin{matrix} AN - AD \\ \frac{x^4}{r^3} - \frac{4xx}{r} + 2r \end{matrix}\right\} = DN$, the Quadruplication.

$$AB : AK :: AF\ (2AN - AD) : AO.$$

$$2r : x :: \frac{2x^4}{r^3} - \frac{6xx}{r} + 2r : \frac{x^5}{r^4} - \frac{3x^3}{rr} + x.$$

And $\left\{\begin{matrix} AO - AE \\ \frac{x^5}{r^4} - \frac{5x^3}{rr} + 5x \end{matrix}\right\} = EO$, the Quintuplication.

And ſo onwards. Now if you would divide an Angle into any Number of Parts, put q for BL, CM, DN, &c. and you'll have $xx - 2rr = qr$ for the Biſection; $xxx - 3rrx = qr^2$ for the Triſection; $xxxx - 4rrxx + 2r^4 = qr^3$ for the Quadriſection; $xxxxx - 5r^2x^3 + 5r^4x = qr^4$ for the Quinquiſection, &c.

PRO-

PROBLEM XVI.

To determine the Position of a Comet's Course [or Way] that moves uniformly in a right Line, [as] BD, from three Observations. [Vide Figure 30.]

SUppose A to be the Eye of the Spectator, B the Place of the Comet in the first Observation, C in the second, and D in the third; the Inclination of the Line BD to the Line AB is to be found. From the Observations therefore there are given the Angles BAC, BAD; and consequently if BH be drawn perpendicular to AB, and meeting AC and AD in E and F, assuming any how AB, there will be given BE and BF, *viz.* the Tangents of the Angles in respect of the Radius AB. Make therefore $AB = a$, $BE = b$, and $BF = c$. Moreover, from the given Intervals [or Distances] of the Observations, there will be given the Ratio of BC to BD, which, if it be made as b to e, and DG be drawn parallel to AC, since BE is to BG in the same Ratio, and BE was call'd b, BG will be $= e$, and consequently $GF = e - c$. Moreover, if you let fall DH perpendicular to BG, by reason of the Triangles ABF and DHF being like, and alike divided by the Lines AE and DG, FE will be $: AB :: FG : HD$, that is, $c - b : a :: e - c : \frac{ae - ac}{c - b} = HD$. Moreover, FE will be $: FB :: FG : FH$, that is, $c - b : c :: e - c : \frac{ce - cc}{c - b} = FH$; to which add BF, or c, and BH will be $= \frac{ce - cb}{c - b}$. Wherefore $\frac{ce - cb}{c - b}$ is to $\frac{ae - ac}{c - b}$ (or $ce - cb$ to $ae - ac$, or $\frac{ce - cb}{e - c}$ to a) as BH to HD; that is, as the Tangent of the Angle HDB, or ABK to the Radius. Wherefore, since a is suppos'd to be the Radius, $\frac{ce - cb}{e - c}$ will be the Tangent of the Angle ABK, and therefore by resolving

[them

[them into an Analogy] 'twill be as $e-c$ to $e-b$, (or *GF* to *GE*) so c (or the Tangent of the Angle *BAF*) to the Tangent of the Angle *ABK*.

Say therefore, as the Time between the first and second Observation to the Time between the first and third, so the Tangent of the Angle *BAE* to a fourth Proportional. Then as the Difference between that fourth Proportional and the Tangent of the Angle *BAF*, to the Difference between the same fourth Proportional and the Tangent of the Angle *BAE*, so the Tangent of the Angle *BAF* to the Tangent of the Angle *ABK*.

PROBLEM XVII.

Rays [of Light] from any shining or lucid Point diverging to a refracting Spherical Surface, to find the Concourse of each of the refracted Rays with the Ax of the Sphere passing thro' that lucid Point. [Vide Figure 31.]

LET *A* be that lucid Point, and *BV* the Sphere, the Axis whereof is *AD*, the Center *C*, and the Vertex *V*; and let *AB* be the incident Ray, and *BD* the refracted Ray; and having let fall to those Rays the Perpendiculars *CE* and *CF*, as also *BG* perpendicular to *AD*, and having drawn *BC*, make $AC=a$, VC or $BC=r$, $CG=x$, and $CD=z$, and *AG* will be $=a-x$, $BG=\sqrt{rr-xx}$, $AB=\sqrt{aa-2ax+rr}$; and by reason of the similar Triangles *ABG* and *ACE*, *CE* will $=\dfrac{a\sqrt{rr-xx}}{\sqrt{aa-2ax+rr}}$. Also $GD=z+x$, $BD=\sqrt{zz+2zx+rr}$; and by reason of the similar Triangles *DBG* and *DCF*, $CF=\dfrac{z\sqrt{rr-xx}}{\sqrt{zz+2zx+rr}}$. Besides, since the Ratio of the Sines of Incidence and Refraction, and consequently of *CE* to *CF*, is given, suppose that Ratio to be as a to f, and $\dfrac{fa\sqrt{rr-xx}}{\sqrt{aa-2ax+rr}}$ will be $=\dfrac{az\sqrt{rr-xx}}{\sqrt{zz+2zx+rr}}$; and multi-

plying

plying cross-ways, and dividing by $a\sqrt{rr-xx}$, $f\sqrt{zz+2zx+rr}$ will be $=z\sqrt{aa-2xa+rr}$, and by squaring and reducing the Terms into Order, $zz=\frac{2ffxz+ffrr}{aa-2ax+rr-ff}$. Then for the given $\frac{ff}{a}$ write p, and q for the given $a+\frac{rr}{a}-p$, and zz will be $=\frac{2pxz+prr}{q-2x}$, and $z=\frac{px+\sqrt{ppxx-2prrx+pqrr}}{q-2x}$. Therefore z is found; that is, the Length of CD, and consequently the Point sought D, where the refracted Ray BD meets with the Axis. Q. E. F.

Here I made the incident Rays to diverge, and fall upon a thicker Medium; but changing what is requisite to be changed, the Problem may be as easily resolved when the Rays converge, or fall from a thicker Medium into a thinner one.

PROBLEM XVIII.

If a Cone be cut by any Plane, to find the Figure of the Section. [Vide Figure 32.]

LET ABC be a Cone standing on a circular Base BC, and IEM its Section sought; and let $KILM$ be any other Section parallel to the Base, and meeting the former Section in HI; and ABC a third Section, perpendicularly bisecting the two former in EH and KL, and the Cone in the Triangle ABC, and producing EH till it meet AK in D; and having drawn EF and DG parallel to KL, and meeting AB and AC in F and G, call $EF=a$, $DG=b$, $ED=c$, $EH=x$, and $HI=y$; and by reason of the similar Triangles EHL, EDG, ED will be $:DG::EH:HL=\frac{bx}{c}$. Then by reason of the similar Triangles DEF, DHK, DE will be $:EF::DH:$ ($c-x$ in the first Figure, and $c+x$ in the second Figure) $HK=\frac{ac\mp ax}{c}$. [Vide Figure 33.] Lastly, since the Section KIL is parallel to the Base, and consequently circular, HK

$HK \times HL$ will be $= HIq$, that is, $\frac{ab}{c}x \mp \frac{ab}{cc}xx = yy$, an Æquation which expresses the Relation between EH (x) and HI (y), that is, between the Axis and the Ordinate of the Section EIM; which Æquation, since it expresses an Ellipse in the first Figure, and an Hyperbola in the second Figure, it is evident, that that Section will be Ellipsical or Hyperbolical.

Now if ED no where meets AK, being parallel to it, then HK will be $= EF$ (a), and thence $\frac{ab}{c}x$ $(HK \times HL)$ $= yy$, an Æquation expressing a Parabola.

Problem XIX.

If the right Line XY be turn'd about the Axis AB, at the Distance CD, with a given Inclination to the Plane DCB, and the Solid PQRUTS, generated by that Circumrotation, be cut by any Plane [as] INQLK, to find the Figure of the Section. [Vide Figure 34.]

LET BHQ, or GHO be the Inclination of the Axis AB to the Plane of the Section; and let L be any Concourse of the right Line XY with that Plane. Draw DF parallel to AB, and let fall the Perpendiculars LG, LF, LM, to AB, DF, and HO, and join FG and MG. And having call'd $CD = a$, $CH = b$, $HM = x$, and $ML = y$, by reason of the given Angle GHO, making $MH : HG :: d : e$, $\frac{ex}{d}$ will $= GH$, and $b + \frac{ex}{d} =$ to GC or FD. Moreover, by reason of the given Angle LDF (viz. the Inclination of the right Line XY to the Plane $GCDF$) putting $FD : FL :: g : h$, $\frac{hb}{g} + \frac{hex}{dg} = FL$, to whose Square add FGq $(DCq$, or $aa)$ and there will come out $GLq = aa + \frac{hhbb}{gg} + \frac{2hhbex}{dgg} + \frac{hheexx}{ddgg}$. Hence subtract MGq $(HMq - HGq$, or $xx - \frac{ee}{dd}xx)$ and there

will

will remain $\frac{aagg+hhbb}{gg}+\frac{2hhbe}{dgg}x+\frac{hhee-ddgg+eegg}{ddgg}$ $xxx(=MLq)=yy$: an Æquation that expresses the Relation between x and y, that is, between HM the Axis of the Section, and ML its Ordinate. And therefore, since in this Æquation x and y ascend only to two Dimensions, it is evident, that the Figure $INQLK$ is a Conick Section. As for Example, if the Angle MHG is greater than the Angle LDF, this Figure will be an Ellipse; but if less, an Hyperbola; and if equal, either a Parabola, or (the Points C and H moreover coinciding) a Parallelogram.

PROBLEM XX.

If you erect AD of a given Length perpendicular to AF, and ED, one Leg of a Square DEF, pass continually thro' the Point D, while the other Leg EF equal to AD slide upon AF, to find the Curve HIC, which the Leg EF describes by its middle Point C. [Vide Figure 35.]

LET EC or $CF=a$, the Perpendicular $CB=y$, $AB=x$, and BF $(\sqrt{aa-yy})$: $BC+CF$ $(y+a)$ :: EF $(2a)$: $EG+GF=(AG+GF)$ or AF. Wherefore $\frac{2ay+2aa}{\sqrt{aa-yy}}$ $(=AF=AB+BF)$ $=x+\sqrt{aa-yy}$. Now, by multiplying by $\sqrt{aa-yy}$ there is made $2ay+2aa=aa-yy+x\sqrt{aa-yy}$, or $2ay+aa+yy=x\times\sqrt{aa-yy}$, and by squaring the Parts, and dividing by $\sqrt{a+y}$, and ordering them, there comes out y^3+3ayy $\begin{matrix}+3aa\\+xx\end{matrix}y\begin{matrix}+a^3\\-axx\end{matrix}=0$.

The same otherwise. [Vide Figure 36.]

On BC take at each End BI, and CK equal to CF, and draw KF, HI, HC, and DF; whereof HC and DF meet AF, and IK in M and N, and upon HC let fall the

the Perpendicular IL; and the Angle K will be $= \frac{1}{2} BCF = \frac{1}{2} EGF = GFD = AMH = MHI = CIL$; and consequently the right-angled Triangles KBF, FBN, HLI, and ILC will be similar. Make therefore $FC = a$, $HI = x$, and $IC = y$; and BN $(2a - y)$ will be : BK (y) :: $LC : LH$ :: CIq. (yy) : HIq (xx), and consequently $2axx - yxx = y^3$. From which Æquation it is easily inferr'd, that this Curve is the Cissoid of the Antients, belonging to a Circle, whose Center is A, and its Radius AH.

PROBLEM XXI.

If a right Line ED of a given Length subtending the given Angle EAD, be so moved, that its Ends D and E always touch the Sides AD and AE of that Angle; let it be propos'd to determine the Curve FCG, which any given Point C in that right Line ED describes. [Vide Figure 37.]

FROM the given Point C draw CB parallel to EA; and make $AB = x$, $BC = y$, $CE = a$, and $CD = b$, and by reason of the similar Triangles DCB, DEA, EC will be : AB :: $CD : BD$; that is, $a : x :: b : BD = \frac{bx}{a}$. Besides, having let fall the Perpendicular CH, by reason of the given Angle DAE, or DBC, and consequently of the given Ratio of the Sides of the right-angled Triangle BCH, you'll have $a : e :: BC : BH$, and BH will be $= \frac{ey}{a}$. Take away this from BD, and there will remain $HD = \frac{bx - ey}{a}$. Now in the Triangle BCH, because of the right Angle BHC, $BCq - BHq = CHq$; that is, $yy - \frac{eeyy}{aa} = CHq$. In like manner, in the Triangle CDH, because of the right Angle CDH, $CDq - CHq$ is $= HDq$; that is, $bb - yy + \frac{eeyy}{aa}$ $\left(= HDq = \frac{bx - ey}{a} q.\right)$

$= bbxx$

Fig. 24.

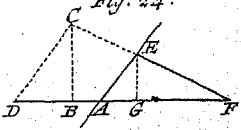


Fig. 25.

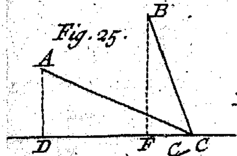


Fig. 26.

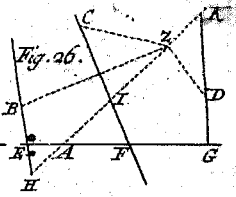


Fig. 27.

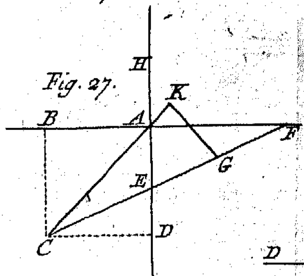


Fig. 28.

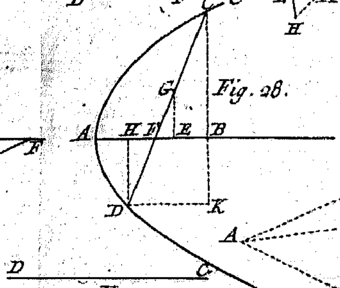


Fig. 30.

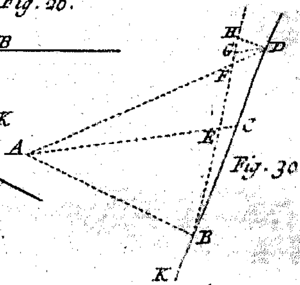


Fig. 29.

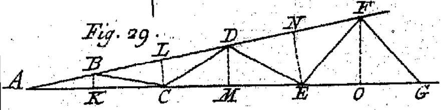


Fig. 31.

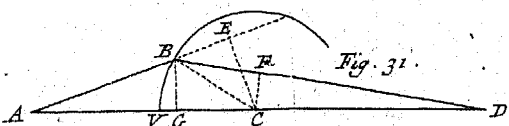


Fig. 32.

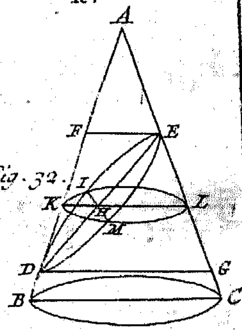


Fig. 34.

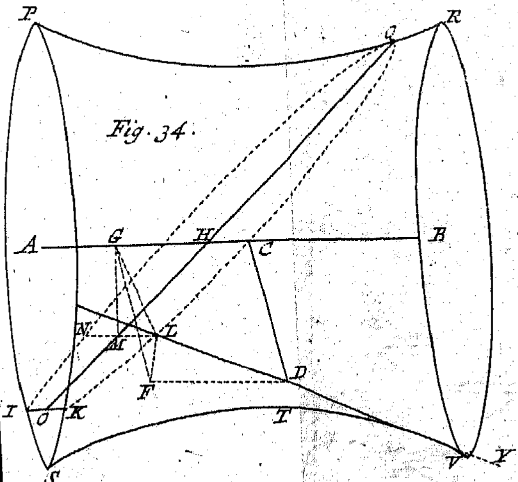
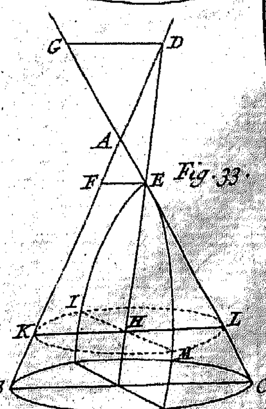


Fig. 33.



$= \frac{bbxx - 2bexy + eeyy}{aa}$; and by Reduction $yy = \frac{2be}{aa} \times xy + \frac{aabb - bbxx}{aa}$. Where, since the unknown Quantities are of two Dimensions, it is evident that the Curve is a Conick Section. Then extracting the Root, you'll have $y = \frac{bex \mp b\sqrt{eexx - aaxx + a^4}}{aa}$. Where, in the Radical Term, the Coefficient of xx is $ee - aa$. But it was $a : e :: BC : BH$; and BC is necessarily a greater Line than BH, *viz.* the Hypothenuse of a right-angled Triangle is greater than the Side of it; therefore a is greater than e, and $ee - aa$ is a negative Quantity, and consequently the Curve will be an Ellipsis.

PROBLEM XXII.

If the Ruler EBD, forming a right Angle, be so moved, that one Leg of it, EB, continually subtends the right Angle EAB, while the End of the other Leg, BD, describes some Curve Line, as FD; to find that Line FD, which the Point D describes. [Vide Figure 38.]

FROM the Point D let fall the Perpendicular DC to the Side AC; and making $AC = x$, and $DC = y$, and $EB = a$, and $BD = b$. In the Triangle BDC, by reason of the right Angle at C, BCq is $= BDq - DCq = bb - yy$. Therefore $BC = \sqrt{bb - yy}$; and $AB = x - \sqrt{bb - yy}$. Besides, by reason of the similar Triangles BEA, DBC, $BD : DC :: EB : AB$; that is, $b : y :: a : x - \sqrt{bb - yy}$; therefore $bx - b\sqrt{bb - yy} = ay$, or $bx - ay = b\sqrt{bb - yy}$. And the Parts being squar'd and duly reduc'd $yy = \frac{2abxy + b^4 - bbxx}{aa + bb}$, and extracting the Root $y = \frac{abx \pm bb\sqrt{aa + bb - xx}}{aa + bb}$. Whence it is again evident, that the Curve is an Ellipse.

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This is so where the Angles EBD and EAB are right; but if those Angles are of any other Magnitude, as long as they are equal, you may proceed thus: [*Vide Figure* 39.] Let fall DC perpendicular to AC as before, and draw DH, making the Angle DHA equal to the Angle HAE, suppose Obtuse, and calling $EB = a$, $BD = b$, $AH = x$, and $HD = y$; by reason of the similar Triangles EAB, BHD, BD will be : $DH :: EB : AB$; that is, $b : y :: a : AB = \frac{ay}{b}$. Take this from AH and there will remain $BH = x - \frac{ay}{b}$. Besides, in the Triangle DHC, by reason of all the Angles given, and consequently the Ratio of the Sides given, assume DH to HC in any given Ratio, suppose as b to e; and since DH is y, HC will be $\frac{ey}{b}$, and $HB \times HC$ will $= \frac{exy}{b} - \frac{aeyy}{bb}$. Lastly, by the 12, 2 *Elem.* in the Triangle BHD, BDq is $= BHq + DHq + 2BH \times HC$; that is, $bb = xx - \frac{2axy}{b} + \frac{aayy}{bb} + yy + \frac{2exy}{b} - \frac{2aeyy}{bb}$. and extracting the Root $x = \frac{ay - ey \pm \sqrt{eeyy - bbyy + bbbb}}{b}$. Where, when b is greater than e, that is, when $ee - bb$ is a negative Quantity, it is again evident, that the Curve is an Ellipse.

PROBLEM XXIII.

Having the Sides and Base of any right-lined Triangle given, to find the Segments of the Base, the Perpendicular, the Area, and the Angles. [Vide Figure 40.]

LET there be given the Sides AC, BC, and the Base AB of the Triangle ABC. Bisect AB in I, and take on it (being produc'd on both Sides) AF and AE equal to AC, and BG and BH equal to BC. Join CE, CF; and from C to the Base let fall the Perpendicular CD. And $ACq - BCq$ will be $= ADq + CDq - CDq - BDq = ADq$

$= ADq - BDq = \overline{AD + BD} \times \overline{AD - BD} = AB \times 2DI$. Therefore $\frac{ACq - BCq}{2AB} = DI$. And $2AB : AC + BC :: AC - BC : DI$. Which is a Theorem for determining the Segments of the Base.

From IE, that is, from $AC - \frac{1}{2}AB$, take away DI, and there will remain $DE = \frac{BCq - ACq + 2AC \times AB - ABq}{2AB}$, that is, $= \frac{\overline{BC + AC - AB} \times \overline{BC - AC + AB}}{2AB}$, or $= \frac{HE \times EG}{2AB}$. Take away DE from FE, or $2AC$, and there will remain $FD = \frac{ACq + 2AC \times AB + ABq - BCq}{2AB}$; that is, $= \frac{\overline{AC + AB + BC} \times \overline{AC + AB - BC}}{2AB}$, or $= \frac{FG \times FH}{2AB}$. And since CD is a mean Proportional between DE and DF, and CE a mean Proportional between DE and EF, and CF a mean Proportional between DF and EF, CD will be $= \frac{\sqrt{FG \times FH \times HE \times EG}}{2AB}$, $CE = \sqrt{\frac{AC \times HE \times EG}{AB}}$, and $CF = \sqrt{\frac{AC \times FG \times FH}{AB}}$. Multiply CD into $\frac{1}{2}AB$, and you'll have the Area $= \frac{1}{4}\sqrt{FG \times FH \times HE \times EG}$. But for determining the Angle A, there come out several Theorems:

1. As $2AB \times AC : HE \times EG$ ($:: AC : DE$) :: Radius : versed Sine of the Angle A.
2. $2AB \times AC : \sqrt{FG \times FH}$ ($:: AC : FD$) :: Radius : versed Cosine of A.
3. $2\overline{AB \times AC} : \sqrt{\overline{FG \times FH} \times \overline{HE \times EG}}$ ($:: AC : CD$) :: Radius : Sine of A.
4. $\sqrt{FG \times FH} : \sqrt{HE \times EG}$ ($:: CF : CE$) :: Radius : Tangent of $\frac{1}{2}A$.
5. $\sqrt{HE \times EG} : \sqrt{FG \times FH}$ ($:: CE : FC$) :: Radius : Cotangent of $\frac{1}{2}A$.

6. $2\sqrt{AB}$

6. $2\sqrt{AB \times AC} : \sqrt{HE \times EG}$ ($:: FE : CE$) :: Radius : Sine of $\frac{1}{2}A$.

7. $2\sqrt{AB \times AC} : \sqrt{FG \times FH}$ ($:: FE : FC$) :: Radius : Cosine of $\frac{1}{2}A$.

PROBLEM XXIV.

In the given Angle PAB having any how drawn the right Lines, BD, PD, in a given Ratio, on this Condition, that BD shall be parallel to AP, and PD terminated at the given Point P in the right Line AP; to find the Locus of the Point D. [Vide Figure 41.]

DRAW CD parallel to AB, and DE perpendicular to AP; and make $AP = a$, $CP = x$, and $CD\,y$, and let BD be to PD in the same Ratio as d to e, and AC or BD will be $= a - x$, and $PD = \frac{ea - ex}{d}$. Moreover, by reason of the given Angle DCE, let the Ratio of CD to CE be as d to f, and CE will be $= \frac{fy}{d}$, and $EP = x - \frac{fy}{d}$. But by reason of the Angles at E being right ones, $CDq - CEq$ will be ($= EDq$) $= PDq - EPq$; that is, $yy - \frac{ffyy}{dd} = \frac{eeaa - 2eeax + eexx}{dd} - xx + \frac{2fxy}{d} - \frac{ffyy}{dd}$, and blotting out on each Side $-\frac{ffyy}{dd}$, and the Terms being rightly dispos'd, $yy = \frac{2fxy}{d} + \frac{eeaa - 2eeax + eexx - ddxx}{dd}$, and extracting the Root

$$y = \frac{fx}{d} \mp \sqrt{\frac{eeaa - 2eeax \begin{smallmatrix} +ee \\ -dd \\ +ff \end{smallmatrix} xx}{dd}}.$$

Where,

Where, since x and y in the last Æquation ascends only to two Dimensions, the Place of the Point D will be a Conick Section, and that either an Hyperbola, Parabola, or Ellipse, as $ee - dd + ff$, (the Co-efficient of xx in the last Æquation) is greater, equal to, or less than nothing.

PROBLEM XXV.

The two right Lines VE and VC being given in Position, and cut any how in C and E by another right Line, PE turning about the Pole, P given also in Position; if the intercepted Line CE be divided into the Parts CD, DE that have a given Ratio to one another, it is propos'd to find the Place of the Point D. [Vide Figure 42.]

DRAW VP, and parallel to it DA, and EB meeting VC in A and B. Make $VP = a$, $VA = x$, and $AD = y$, and since the Ratio of CD to DE is given, or conversely of CD to CE, that is, the Ratio of DA to EB, let it be as d to e, and EB will be $= \frac{ey}{d}$. Besides, since the Angles EVB, EVP are given, and consequently the Ratio of EB to VB, let that Ratio be as e to f, and VB will be $= \frac{fy}{d}$. Lastly, by reason of the similar Triangles CEB, CDA, CPV, $EB : CB :: DA : CA :: VP : VC$, and by Composition $EB + VP : CB + VC :: DA + VP : CA + VC$; that is, $\frac{ey}{d} + a : \frac{fy}{d} :: y + a : x$, and multiplying together the Means and Extremes $eyx + dax = fyy + fay$.

Where, since the indefinite Quantities x and y ascend only to two Dimensions, it follows, that the Curve VD, in which the Point D is always found, is a Conick Section, and that an Hyperbola, because one of the indefinite Quantities, *viz.* x is only of one Dimension, and the Term exy is multiply'd by the other indefinite one y.

PROBLEM XXVI.

If two right Lines, AC and AB, in any given Ratio, are drawn from the two Points A and B given in Position, to a third Point C, to find the Place of C, the Point of Concourse. [Vide Figure 43.]

JOIN AB, and let fall to it the Perpendicular CD; and making $AB=a$, $AD=x$, $DC=y$, AC will be $=\sqrt{xx+yy}$, $BD=x-a$, and BC $(=\sqrt{BDq+DCq})$ $=\sqrt{xx-2ax+aa+yy}$. Now since there is given the Ratio of AC to BC, let that be as d to e; and the Means and Extremes being multiply'd together, you'll have $e\sqrt{xx+yy}=d\sqrt{xx-2ax+aa+yy}$, and by Reduction $\sqrt{\frac{ddaa-2ddax}{ee-dd}-xx}=y$. Where, since x is Negative, and affected only by Unity, and also the Angle ADC a right one, it is evident, that the Curve in which the Point C is plac'd is a Circle, *viz.* in the right Line AB take the Points E and F, so that $d:e::AE:BE::AF:BF$, and EF will be the Diameter of this Circle.

And hence from the Converse this Theorem comes out, that in the Diameter of any Circle EF being produc'd, having given any how the two Points A and B on this Condition, that $AE:AF::BE:BF$, and having drawn from these Points the two right Lines AC and BC, meeting the Circumference in any Point C; AC will be to BC in the given Ratio of AE to BE.

PROBLEM XXVII.

To find the Point D, from which three right Lines DA, DB, DC, let fall perpendicular to so many other right Lines AE, BF, CF, given in Position, shall obtain a given Ratio to one another. [Vide Figure 44.]

OF the right Lines given in Position, let us suppose BF be produc'd, as also its Perpendicular BD, till they meet the rest AE and CF, viz. BF in E and F, and BD in H and G. Now let $EB = x$, and $EF = a$; and BF will be $= a - x$. But since, by reason of the given Position of the right Lines EF, EA, and FC, the Angles E and F, and consequently the Proportions of the Sides of the Triangles EBH and FBG are given. Let EB be to BH as d to e; and BH will be $= \frac{ex}{d}$, and EH ($= \sqrt{EBq + BHq} = \sqrt{xx + \frac{eexx}{dd}}$, that is, $\frac{x}{d} \times \sqrt{dd + ee}$. Let also BF be to BG as d to f; and BG will be $= \frac{fa - fx}{d}$, and FG ($= \sqrt{BFq + BGq}$) $=$

$$\sqrt{\frac{aadd - 2axdd + xxdd + ffaa - 2ffax + ffxx}{dd}},$$

that is, $= \frac{a - x}{d} \sqrt{dd + ff}$. Besides, make $BD = y$, and HD will be $= \frac{ex}{d} - y$, and $GD = \frac{fa - fx}{d} - y$; and so, since AD is : HD (: : $EB : EH$) : : $d : \sqrt{dd + ee}$, and $DC : GD$ (: : $BF : FG$) : : $d : \sqrt{dd + ff}$, AD will be $= \frac{ex - dy}{\sqrt{dd + ee}}$, and $DC = \frac{fa - fx - dy}{\sqrt{dd + ff}}$. Lastly, by reason of the given Proportions of the Lines BD, AD, DC, let $BD : AD : : \sqrt{dd + ee} : h - d$, and $\frac{hy - dy}{\sqrt{dd + ee}}$ will be

 (=

$(=AD) = \frac{ex - dy}{\sqrt{dd+ee}}$, or $hy = ex$. Let also $BD : DC :: \sqrt{dd+ff} : k - d$, and $\frac{ky - dy}{\sqrt{dd+ff}}$ will be $(= DC) = \frac{fa - fx - dy}{\sqrt{dd+ff}}$, or $ky = fa - fx$. Therefore $\frac{ex}{h}$ $(= y) = \frac{fa - fx}{k}$; and by Reduction $\frac{fha}{ek + fh} = x$. Wherefore take $EB : EF :: h : \frac{ek}{f} + h$, then $BD : EB :: e : h$, and you'll have the Point sought D.

PROBLEM XXVIII.

To find the Point D, from which three right Lines DA, DB, DC, drawn to the three Points, A, B, C, shall have a given Ratio among themselves. [Vide Figure 45.]

Of the given three Points join any two of them, as suppose A and C, and let fall the Perpendicular BE from the third B, to the Line that conjoins A and C, as also the Perpendicular DF from the Point sought D; and making $AE = a$, $AC = b$, $EB = c$, $AF = x$, and $FD = y$; and ADq will be $= xx + yy$. $FC = b - x$. CDq $(= FCq + FDq) = bb - 2bx + xx + yy$. $EF = x - a$, and BDq $(= EFq + \overline{EB + FD}^q) = xx - 2ax + aa + cc + 2cy + yy$. Now, since AD is to CD in a given Ratio, let it be as d to e; and CD will be $= \frac{e}{d}\sqrt{xx+yy}$. Since also AD is to BD in a given Ratio, let that be as d to f, and BD will be $= \frac{f}{d}\sqrt{xx+yy}$. And, consequently $\frac{eexx + eeyy}{dd}$ will be $(= CDq) = bb - 2bx + xx + yy$, and $\frac{ffxx + ffyy}{dd}$ $(= BDq) = xx - 2ax + aa + cc + 2cy + yy$.

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In which if, for Abbreviation sake, you write p for $\frac{dd-ee}{d}$, and q for $\frac{dd-ff}{d}$, there will come out $bb-2bx+\frac{p}{d}xx+\frac{p}{d}yy=0$, and $aa+cc-2ax+2cy+\frac{q}{d}xx+\frac{q}{d}yy=0$. And by the former you have $\frac{2bqx-bbq}{p}=\frac{q}{d}xx+\frac{q}{d}yy$. Wherefore, in the latter, for $\frac{q}{d}xx+\frac{q}{d}yy$, write $\frac{2bqx-bbq}{p}$, and there will come out $\frac{2bqx-bbq}{p}+aa+cc-2ax+2cy=0$. Again, for Abbreviation sake, write m for $a-\frac{bq}{p}$, and $2cn$ for $\frac{bbq}{p}-aa-cc$, and you'll have $2mx+2cn=2cy$, and the Terms being divided by $2c$, there arises $\frac{mx}{c}+n=y$. Wherefore, in the Æquation $bb-2bx+\frac{p}{d}xx+\frac{p}{d}yy=0$, for yy write the Square of $\frac{mx}{c}+n$, and you'll have $bb-2bx+\frac{p}{d}xx+\frac{pmm}{dcc}xx+\frac{2pmn}{dc}x+\frac{pnn}{d}=0$. Where, lastly, if, for Abbreviation sake, you write $\frac{b}{r}$ for $\frac{p}{d}+\frac{pmm}{dcc}$, and $\frac{sb}{r}$ for $b-\frac{pmn}{dc}$, you'll have $xx=2sx-rb-\frac{pnnr}{bd}$, and having extracted the Root $x=s\pm\sqrt{ss-rb-\frac{pnnr}{bd}}$, and having found x, the Æquation $\frac{mx}{c}+n=y$ will give y; and from x and y given, that is, AF and FD, the given Point D is determin'd.

PROBLEM XXIX.

To find the Triangle ABC, whose three Sides AB, AC, BC, and its Perpendicular DC are in Arithmetical Progression. [Vide Figure 46.]

MAKE $AC = a$, $BC = x$, and DC, the least Line, will be $= 2x - a$, and AB, the greatest, will be $= 2a - x$. Also AD will $(= \sqrt{ACq - DCq}) = \sqrt{4ax - 4xx}$, and BD $(= \sqrt{BCq - DCq} = \sqrt{4ax - 3xx - aa}$. And so again, $AB = \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$. Wherefore $2a - x = \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$, or $2a - x - \sqrt{4ax - 4xx} = \sqrt{4ax - 3xx - aa}$. And the Parts being squar'd, $4aa - 3xx - 4a + 2x \times \sqrt{4ax - 4xx} = 4ax - 3xx - aa$, or $5aa - 4ax = 4a - 2x \times \sqrt{4ax - 4xx}$. And the Parts being again squar'd, and the Terms rightly dispos'd, $16x^4 - 80ax^3 + 144aaxx - 104a^3x + 25a^4 = 0$. Divide this Æquation by $2x - a$, and there will arise $8x^3 - 36axx + 54aax - 25a^3 = 0$, an Æquation by the Solution whereof x is given from a, being any how assum'd. a and x being had, make a Triangle, whose Sides shall be $2a - x$, a and x, and a Perpendicular let fall upon the Side $2a - x$, will be $2x - a$.

If I had made the Difference of the Sides of the Triangle to be d, and the Perpendicular to be x, the Work would have been something neater; this Æquation at last coming out, *viz.* $x^3 = 24ddx - 48d^3$.

PROBLEM XXX.

To find a Triangle ABC, whose three Sides AB, AC, BC, and the Perpendicular CD shall be in a Geometrical Progression.

MAKE $AC = x$, $BC = a$, and AB will be $= \frac{xx}{a}$; And $CD = \frac{aa}{x}$. And AD $(= \sqrt{ACq - CDq}) = \sqrt{xx - \frac{a^4}{xx}}$; and BD $(= \sqrt{BCq - DCq}) = \sqrt{aa - \frac{a^4}{xx}}$, and consequently $\frac{xx}{a}$ $(= AB) = \sqrt{xx - \frac{a^4}{xx}} + \sqrt{aa - \frac{a^4}{xx}}$, or $\frac{xx}{a} - \sqrt{aa - \frac{a^4}{xx}} = \sqrt{xx - \frac{a^4}{xx}}$; and the Parts of the Æquation being squar'd, $\frac{x^4}{aa} - \frac{2xx}{a} \times \sqrt{aa - \frac{a^4}{xx}} + aa - \frac{a^4}{xx} = xx - \frac{a^4}{xx}$, that is, $x^4 - aaxx + a^4 = 2aax\sqrt{xx - aa}$. And the Parts being again squar'd, $x^8 - 2aax^6 + 3a^4x^4 - 2a^6xx + a^8 = 4a^4x^4 - 4a^6xx$. That is, $x^8 - 2aax^6 - a^4x^4 + 2a^6xx + a^8 = 0$. Divide this Æquation by $x^4 - aaxx - a^4$, and there will arise $x^4 - aaxx - a^4$. Wherefore x^4 is $= aaxx + a^4$. And extracting the Root $xx = \frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}$, or $x = a\sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$. Take therefore a, or BC, of any Length, and make $BC : AC :: AC : AB :: 1 : \sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$, and the Perpendicular DC of a Triangle ABC made of these Sides, will be to the Side BC in the same Ratio.

The same otherwise. [Vide Figure 47.]

Since $AB : AC :: BC : DC$. I say the Angle ACB is a right one. For if you deny it, draw CE, making the Angle ECB a right one.

There-

Therefore the Triangles BCE, DBC are similar by 8, 6 *Elem.* and consequently $EB : EC :: BC : DC$, that is, $EB : EC :: AB : AC$. Draw AF perpendicular to CE, and by reason of the parallel Lines AF, BC, EB will be $: EC :: AE : FE :: (EB + AE)\ AB : (EC + FE)\ FC$. Therefore by 9, 5 *Elem.* AC is $= FC$, that is, the Hypothenuse of a right-angled Triangle is equal to the Side, contrary to the 19, 1 *Elem.* Therefore the Angle ECB is not a right one; wherefore it is necessary ACB should be a right one. Therefore $ACq + BCq = ABq$. But $ACq = AB \times BC$, therefore $AB \times BC + BCq = ABq$, and extracting the Root $AB = \frac{1}{2}BC + \sqrt{\frac{5}{4}BCq}$. Wherefore take $BC : AB :: 1 : \frac{1 + \sqrt{5}}{2}$, and AC a mean Proportional between BC and AB, and $AB : AC :: BC : DC$ will be continually proportional to a Triangle made of these Sides.

PROBLEM XXXI.

To make the Triangle ABC upon the given Base AB, whose Vertex C shall be in the right Line EC given in Position, and the Base an Arithmetical Mean between the Sides. [Vide Figure 48.]

LET the Base AB be bisected in F, and produc'd till it meet the right Line EC in E, and let fall to it the Perpendicular CD; and making $AB = a$, $FE = b$, and $BC - AB = x$, BC will be $= a + x$, $AC = a - x$; and by the 13, 2 *Elem.* $BD \left(= \frac{BCq - ACq + ABq}{2AB}\right) = 2x + \frac{1}{2}a$. And consequently, $FD = 2x$, $DE = b + 2x$, and $CD \left(= \sqrt{CBq - BDq}\right) = \sqrt{\frac{3}{4}aa - 3xx}$. But by reason of the given Positions of the right Lines CE and AB, the Angle CED is given; and consequently the Ratio of DE to CD, which, if it be put as d to e, will give the Proportion $d : e :: b + 2x : \sqrt{\frac{3}{4}aa - 3xx}$. Whence the Means and Extremes being multiply'd by each other, there arises the Æquation $eb + 2ex = d\sqrt{\frac{3}{4}aa - 3xx}$, the Parts whereof being squar'd and rightly order'd, you have $xx =$

$\frac{3}{4}d^2a^2$

Tab. III. p: 134.

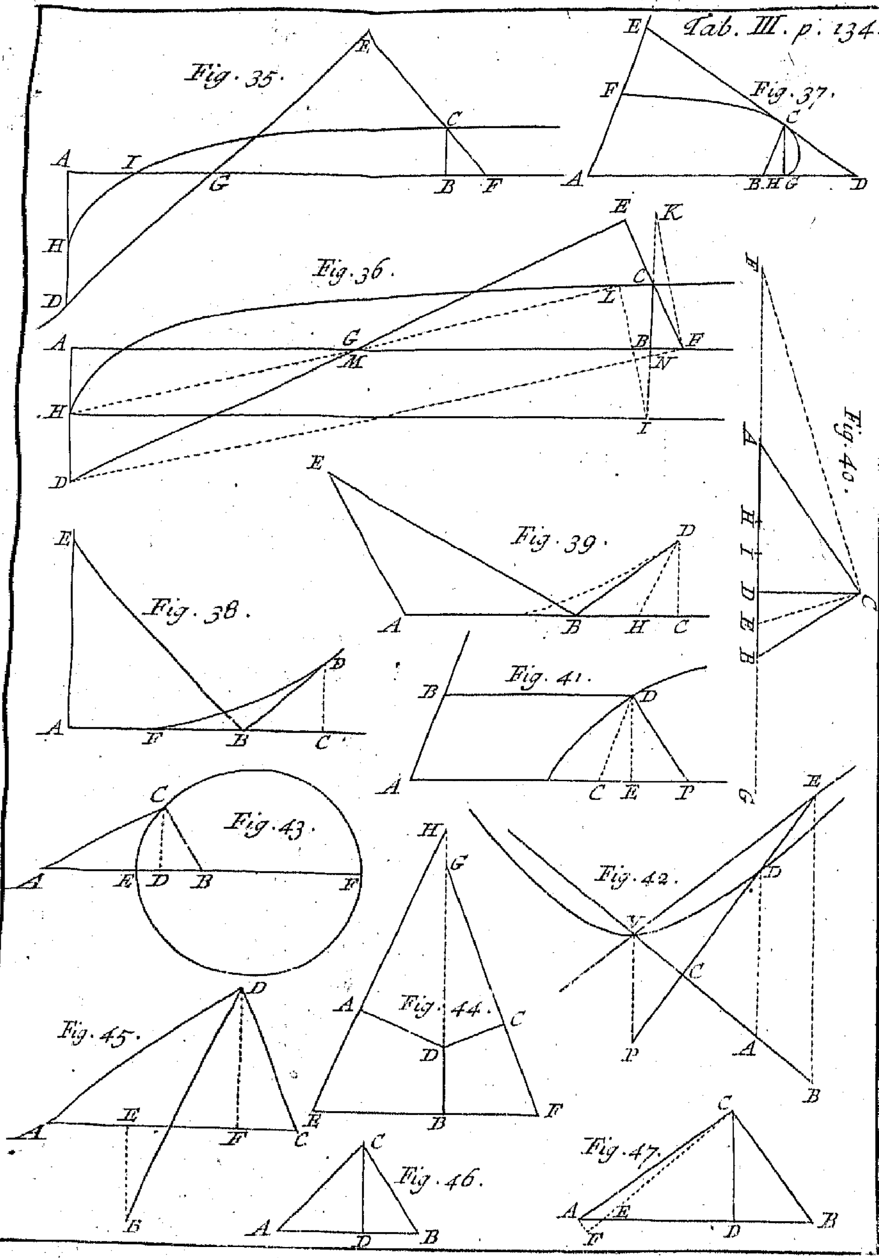
Fig. 35.
Fig. 36.
Fig. 37.
Fig. 38.
Fig. 39.
Fig. 40.
Fig. 41.
Fig. 42.
Fig. 43.
Fig. 44.
Fig. 45.
Fig. 46.
Fig. 47.

$\frac{\frac{3}{4}d^2a^2 - eebb - 4eebx}{4ee + 3dd}$, and the Root being extracted $x = \frac{-2eeb + d\sqrt{3eeaa - 3eebb + \frac{9}{4}ddaa}}{4ee + 3dd}$. But x being given, there is given $BC = a + x$, and $AC = a - x$.

PROBLEM XXXII.

Having the three right Lines AD, AE, BF, given by Position, to draw a fourth DF, whose Parts DE and EF, intercepted by the former, shall be of given Lengths. [Vide Figure 49.]

LET fall EG perpendicular to BF, and draw EC parallel to AD, and the three right Lines given by Position meeting in A, B, and H, make $AB = a$, $BH = b$, $AH = c$. $ED = D$, $EF = e$, and $HE = x$. Now, by reason of the similar Triangles ABH, ECH, $AH : AB :: HE : EC = \frac{ax}{c}$, and $AH : HB :: HE : CH = \frac{bx}{c}$. Add HB, and there comes $CB = \frac{bx + bc}{c}$. Moreover, by reason of the similar Triangles FEC, FDB, ED is : $CB :: EF : CF = \frac{ebx + ebc}{dc}$. Lastly, by the 12 and 13, 2 *Elem.* you have $\frac{ECq - EFq}{2FC} + \frac{1}{2}FC \ (= CG) = \frac{HEq - ECq}{2CH} - \frac{1}{2}CH$; that is,

$$\frac{\frac{aaxx}{cc} - ee}{\frac{2ebx + 2ebc}{dc}} + \frac{ebx + ebc}{2dc} = \frac{xx - \frac{aaxx}{cc}}{\frac{2bx}{c}} - \frac{bx}{2c}.$$ Or

$$\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebx}{d} + \frac{ebc}{d} = \frac{ccx - aax - bbx}{b}.$$

Here, for Abbreviation sake, for $\frac{cc - aa - bb}{b} - \frac{eb}{d}$ write

m,

m, and you'll have $\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebc}{d} = mx$; and all the Terms being multiply'd by $x + c$, there will come out $\frac{aadxx - eedc}{eb} - \frac{ebcx}{d} + \frac{ebcc}{d} = mxx + mcx$. Again, for $\frac{aad}{eb} - m$ write p, and for $mc + \frac{ebc}{d}$ write $2pq$, and for $-\frac{ebcc}{d} + \frac{eedcc}{eb}$ write prr, and xx will become $= 2qx + rr$, and $x = q \pm \sqrt{qq + rr}$. Having found x or HE, draw EC parallel to AB, and take $FC : BC :: e : d$, and having drawn FED, it will satisfy the Conditions of the Question.

PROBLEM XXXIII.

To a Circle described from the Center C, and with the Radius CD, to draw a Tangent DB, the Part whereof PB placed between the right Lines given by Position, AP and AB shall be of a given Length. [Vide Figure 50.]

FROM the Center C to either of the right Lines given by Position, as suppose to AB, let fall the Perpendicular CE, and produce it till it meets the Tangent DB in H. To the same AB let fall also the Perpendicular PG, and making $EA = a$, $EC = b$, $CD = c$, $BP = d$, and $PG = x$; by reason of the similar Triangles PGB, CDH, you'll have $GB\ (\sqrt{dd - xx}) : PB :: CD : CH = \frac{cd}{\sqrt{dd - xx}}$. Add EC, and you'll have $EH = b + \frac{cd}{\sqrt{dd - xx}}$. Moreover, PG is $: GB :: EH : EB = \frac{b}{x}\sqrt{dd - xx} + \frac{cd}{x}$. Moreover, because of the given Angle PAG, there is given the Ratio of PG to AG, which being made as e to f, AG will $= \frac{fx}{e}$. Add EA and BG, and you'll have, lastly, $EB = a$

$+\frac{fx}{e}+\sqrt{dd-xx}$. Therefore $\frac{cd}{x}+\frac{b}{x}\sqrt{dd-xx}=a+\frac{fx}{e}+\sqrt{dd-xx}$, and by Transposition of the Terms, $a+\frac{fx}{e}-\frac{cd}{x}=\frac{b-x}{x}\sqrt{dd-xx}$. And the Parts of the Æquation being squar'd, $aa+\frac{2afx}{e}-\frac{2acd}{x}+\frac{ffxx}{ee}-\frac{2cdf}{e}+\frac{ccdd}{xx}=\frac{bbdd}{xx}-bb-\frac{2bdd}{x}+2bx+dd-xx$. And by a due Reduction

$$x^4+\frac{\begin{matrix}+2aef\\-2bee\end{matrix}\,x^3\;\begin{matrix}+aaee\\+bbee\\-ddee\\-2cdef\end{matrix}\,xx\;\begin{matrix}+2bddee\\-2acdee\end{matrix}\,x\;\begin{matrix}+ccddee\\-bbddee\end{matrix}}{ee+ff}=0.$$

Problem XXXIV.

If a lucid Point, [as] *A, dart forth Rays towards* [or upon] *a refracting plain Surface,* [as] *C, D; to find the Ray AC, whose refracted* [Part] *CB strikes the given Point B.* [Vide Figure 51.]

FROM that lucid Point let fall the Perpendicular *AD* to the refracting Plane, and let the refracted Ray *BC* meet with it, being produc'd out on both Sides, in *E*; and a Perpendicular let fall from the Point *B* in *F*, and draw *BD*; and making $AD=a$, $DB=b$, $BF=c$, $DC=x$, make the Ratio of the Sines of Incidence and Refraction, that is, of the Sines of the Angles *CAD*, *CED*, to be *d* to *e*, and since *EC* and *AC* (as is known) are in the same Ratio, and *AC* is $\sqrt{aa+xx}$, *EC* will be $=\frac{d}{e}\sqrt{aa+xx}$. Besides, *ED* ($=\sqrt{ECq-CDq}$) $=\sqrt{\frac{ddaa+ddxx}{ee}-xx}$, and $DF=\sqrt{bb-cc}$, and $EF=\sqrt{bb-cc}+$

 $\sqrt{ddaa}$

$\sqrt{\frac{ddaa+ddxx}{ee}-xx}$. Lastly, because of the similar Triangles ECD, EBF, $ED:DC::EF:FB$, and multiplying the Values of the Means and Extremes into one another, $c\sqrt{\frac{ddaa+ddxx}{ee}-xx}=x\sqrt{bb-cc}+x\times\sqrt{\frac{ddaa+ddxx}{ee}-xx}$, or $\overline{c-x}\sqrt{\frac{ddaa+ddxx}{ee}-xx}$ $=x\sqrt{bb-cc}$, and the Parts of the Æquation being squar'd and duly dispos'd [into Order],

$$x^4-2cx^3\;\begin{matrix}+ddcc\\+ddaa\end{matrix}\;xx-2ddaacx+ddaacc \atop {-eebb}$$

$$x^4-2cx^3=\frac{\begin{matrix}+ddcc\\+ddaa\\-ccbb\end{matrix}\;xx-2ddaacx+ddaacc}{dd-ee}=0.$$

PROBLEM XXXV.

To find the Locus or Place of the Vertex of a Triangle D, whose Base AB is given, and the Angles at the Base DAB, DBA, have a given Difference. [Vide Figure 52.]

WHERE the Angle at the Vertex, or (which is the same Thing) where the Sum of the Angles at the Base is given, 29. 3. *Euclid.* has taught [us], that the Locus [or Place] of the Vertex is in the Circumference of a Circle; but we have propos'd the finding the Place when the Difference of the Angles at the Base is given. Let the Angle DBA be greater than the Angle DAB, and let ABF be their given Difference, the right Line BF meeting AD in F. Moreover, let fall the Perpendicular DE to BF, as also DC perpendicular to AB, and meeting BF in G. And making $AB=a$, $AC=x$, and $CD=y$, BC will be $=a-x$. Now since in the Triangle BCG there are given all the Angles, there will be given the Ratio of the Sides BC and GC, let that be as d to a, and CG will $=\frac{aa-ax}{d}$; take away this from DC, or y, and there will remain $DG=\frac{dy-aa+ax}{d}$. Besides, because of the similar Triangles

gles BGC, and DGE, $BG : BC :: DG : DE$. And in the Triangle BGC, $a : d :: CG : BC$. And consequently $aa : dd :: CGq : BCq$, and by compounding $aa + dd : dd :: BGq : BCq$, and extracting the Roots $\sqrt{aa+dd} : d$ (: : $BG : BC$) : : $DG : DE$. Therefore $DE = \frac{dy - aa + ax}{\sqrt{aa+dd}}$. Moreover, since the Angle ABF is the Difference of the Angles BAD and ABD, and consequently the Angles BAD and FBD are equal, the right-angled Triangles CAD and EBD will be similar, and consequently the Sides proportional [or] $DA : DC :: DB : DE$. But $DC = y$. DA ($= \sqrt{ACq + DCq}$) $= \sqrt{xx+yy}$. DB ($= \sqrt{BCq + DCq}$) $= \sqrt{aa - 2ax + xx + yy}$, and above DE was $= \frac{dy - aa + ax}{\sqrt{aa+dd}}$. Wherefore $\sqrt{xx+yy} : y :: \sqrt{aa - 2ax + xx + yy} : \frac{dy - aa + ax}{\sqrt{aa+dd}}$, and the Squares of the Means and Extremes being multiply'd by each other

$$aayy - 2axyy + xxyy + y^4 = \frac{ddxxyy + ddy^4 - 2aadxxy}{aa+dd}$$
$$\frac{-2aady^3 + 2adyx^3 + 2adxy^3 + a^4x^2 + a^4yy - 2a^3x^3}{aa+dd}$$
$$\frac{-2a^3xyy + aax^4 + a^2x^2y^2}{aa+dd}.$$

Multiply all the Terms by $aa + dd$, and reduce those Terms that come out into due Order, and there will arise

$$x^4 \begin{matrix} -2a \\ +\frac{2d}{a}y \end{matrix} x^3 \begin{matrix} -2dy \\ +aa \end{matrix} xx \begin{matrix} +\frac{2d}{a}y^3 \\ +\frac{2dd}{a}yy \end{matrix} x \begin{matrix} -ddyy \\ -2dy^3 \\ -y^4 \end{matrix} = 0.$$

Divide this Æquation by $xx - ax \begin{matrix} +dy \\ +yy \end{matrix}$, and there will arise $xx \begin{matrix} -a \\ +\frac{2d}{a} \end{matrix} x \begin{matrix} -yy \\ -dy \end{matrix} = 0$; there come out therefore two Æquations in the Solution of this Problem: The first, $xx - ax \begin{matrix} +dy \\ +yy \end{matrix} = 0$. is in a Circle, *viz.* the Place of the Point D, where the Angle FBD is taken on the other Side of the right

right Line BF than what is describ'd in the Figure, the Angle ABF being the Sum of the Angles DAB and DBA at the Base, and so the Angle ADB at the Vertex being given. The last, *viz.* $xx \begin{matrix}-a\\ +\frac{2d}{a}y\end{matrix} x \begin{matrix}-yy\\ -dy\end{matrix} = 0$, is an Hyperbola, the Place of the Point D, where the Angle FBD obtains the [same] Situation from the right Line BF, which we describ'd in the Figure; that is, so that the Angle ABF may be the Difference of the Angles DAB, DBA, at the Base. But this is the Determination of the Hyperbola: Bisect AB in P; draw PQ, making the Angle BPQ equal to half the Angle ABF: To this draw the Perpendicular PR, and PQ and PR will be the Asymptotes of this Hyperbola, and B a Point through which the Hyperbola will pass.

Hence arises this *Theorem*. Any Diameter, as AB, of a right-angled Hyperbola, being drawn, and having drawn the right Lines AD, BD, AH, BH from it's Ends to any two Points D and H of the Hyperbola; these right Lines will make equal Angles DAH, DBH at the Ends of the Diameter.

The same after a shorter Way. [Vide Figure 53.]

I laid down a Rule about the most commodious Election of Terms to proceed with in the Calculus [of Problems] where there happens any Ambiguity in the Election [of such Terms]. Here the Difference of the Angles at the Base is indifferent in respect to both [or either of the] Angles; and in the Construction of the Scheme, it might equally have been added to the lesser Angle DAB, by drawing from A a right Line parallel to BF, or subtracted from the greater Angle DBA, by drawing the right Line BF. Wherefore I neither add nor subtract it, but add half of it to one of the Angles, and subtract half of it from the other. Then since it is also doubtful whether AC or BC must be made Use of for the indefinite Term whereon the Ordinate DC stands, I use neither of them; but I bisect AB in P, and I make use of PC; or rather, having drawn MPQ making on both Sides the Angles APQ, BPM equal to half the Difference of the Angles at the Base, so that it, with the right Lines AD, BD, may make the Angles DQP, DMP

equal;

equal; I let fall to MQ the Perpendiculars AR, BN, DO, and I use DO for the Ordinate, and PO for the indefinite Line it stands on. I make therefore $PO = x$, $DO = y$, AR or $BN = b$, and PR or $PN = c$. And by reason of the similar Triangles BNM, DOM, BN will be : DO :: MN : MO. And by Division [as in the 5th of *Euclid*] $DO - BN$, $(y - b) : DO\ (y) :: MO - MN\ (ON$ or $c - x) : MO$. Wherefore $MO = \frac{cy - xy}{y - b}$. In like Manner on the other Side, by reason of the similar Triangles ARQ, DOQ, AR will be : DO :: RQ : QO, and by Composition $DO + AR\ (y + b) : DO\ (y) :: QO + RQ\ (OR$ or $c + x) : QO$. Wherefore $QO = \frac{cy + xy}{y + b}$. Lastly, by reason of the equal Angles DMQ, DQM, MO and QO are equal, that is, $\frac{cy - xy}{y - b} = \frac{cy + xy}{y + b}$. Divide all by y, and multiply by the Denominators, and there will arise $cy + cb - xy - xb = cy - cb + xy - xb$, or $cb = xy$, an Æquation that expresses (as is commonly known) the Hyperbola.

Moreover, the Locus, or Place of the Point D might have been found without an Algebraick Calculus; for from what we have said above, $DO - BN : ON :: DO : MO\ (QO) :: DO + AR : OR$. That is, $DO - BN : DO + BN :: ON : OR$. And mixtly, $DO : BN :: \frac{ON + OR}{2}\ (NP) : \frac{OR - ON}{2}\ (OP)$. And consequently, $DO \times OP = BN \times NP$.

PROBLEM XXXVI.

To find the Locus or Place of the Vertex of a Triangle whose Base is given, and one of the Angles at the Base differs by a given Angle from [being] double of the other.

IN the last Scheme of the former Problem, let ABD be that Triangle, AB its Base bisected in P, APQ or BPM half of the given Angle, by which DBA exceeds the double of the Angle DAB; and the Angle DMQ will be be double of the Angle DQM. To PM let fall the Perpendiculars AR, BN, DO, and bisect the Angle DMQ by the right Line MS meeting DO in S; and the Triangles DOQ, SOM will be similar; and consequently $OQ : OM :: OD : OS$, and dividing $OQ - OM : OM :: SD : OS ::$ (by the 3. of the 6th *Elem.*) $DM : OM$. Wherefore the 9. of the 5th *Elem.*) $OQ - OM = DM$. Now making $PO = x$, $OD = y$, AR or $BN = b$, and PR or $PN = c$, you'll have, as in the former Problem, $OM = \frac{cy - xy}{y - b}$, and $OQ = \frac{cy + xy}{y + b}$, and consequently $OQ - OM = \frac{2bcy - 2xyy}{yy - bb}$. Make now $DOq + OMq = DMq$, that is, $yy + \frac{cc - 2cx + xx}{yy - 2by + bb} yy = \frac{4bbcc - 8bcxy + 4xxyy}{y^4 - 2bbyy + b^4} yy$, or $yy + \frac{cc - 2cx + xx}{\overline{y - b} \times \overline{y - b}} = \frac{4bcc - 8bcxy + 4xxyy}{\overline{y-b} \times \overline{y-b} \times \overline{y+b} \times \overline{y+b}}$; and by due Reduction there will at length arise

$$y^4 * \begin{matrix} +cc \\ -2bb \\ -2cx \\ -3xx \end{matrix} yy \begin{matrix} +2bxx \\ +4bcx \\ +2bcc \end{matrix} y \begin{matrix} +b^4 \\ -3bbcc \\ -2bbcx \\ +bbxx \end{matrix} = 0.$$

Which gives the Relation of the Curve: Which becomes an Hyperbola when the Angle BPM (vanishes, or) becomes nothing; or, which is the same Thing, when one of the Angles at the Base DBA is double of the other DAB. For then BN or b vanishing, the Æquation will become $yy = 3xx + 2cx - cc$.

And from the Construction of this Æquation there comes this *Theorem*. [*Vide Figure* 54.] If from the Center C, the Asymptotes being CS, CT, containing the Angle SCT of 120 Degrees, you describe any Hyperbola, as DV, whose Semi-Axis are CV, CA; produce CV to B, so that VB shall $= VC$, and from A and B you draw any how the right Lines AD, BD, meeting at the Hyperbola; the Angle BAD will be half the Angle ABD, but a third Part of the Angle ADE, which the right Line AD comprehends together with BD produc'd. This is to be understood of an Hyperbola that passes thro' the Point V. Now if the two right Lines Ad and Bd, drawn from the same Points A and B, meet in the conjugate Hyperbola that passes through A, then

then of those two external Angles of the Triangle at the Base, that at B will be double of that at A.

PROBLEM XXXVII.

To describe a Circle through two given Points that shall touch a right Line given by Position. [Vide Figure 55.]

LET A and B be the two Points, and EF the right Line given by Position, and let it be requir'd to describe a Circle ABE through those Points which shall touch that right Line FE. Join AB and bisect it in D. Upon D erect the Perpendicular DF meeting the right Line FE in F, and the Center of the Circle will fall upon this last drawn Line DF, as suppose in C. Join therefore CB; and on FE let fall the Perpendicular CE, and E will be the Point of Contact, and CB and CE equal, as being Radii of the Circle sought. Now since the Points A, B, D, and F, are given, let $DB = a$, and $DF = b$; and seek for DC to determine the Center of the Circle, which therefore call x. Now in the Triangle CDB, because the Angle at D is a right one, you have $\sqrt{DBq + DCq}$, that is, $\sqrt{aa + xx} = CB$. Also $DF - DC$, or $b - x = CF$. And since in the right-angled Triangle CFE the Angles are given, there will be given the Ratio of the Sides CF and CE. Let that be as d to e; and CE will be $= \frac{e}{d} \times CF$, that is, $= \frac{eb - ex}{d}$. Now put [or make] CB and CE (the Radii of the Circle sought) equal to one another, and you'll have the Æquation $\sqrt{aa + xx} = \frac{eb - ex}{d}$. Whose Parts being squar'd and multiply'd by dd, there arises $aadd + ddxx = eebb - 2eebx + eexx$; or $xx = \frac{-2eebx - aadd + eebb}{dd - ee}$.

And extracting the Root $x = \frac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$.

Therefore the Length of DC, and consequently the Center C is found, from which a Circle is to be describ'd through the Points A and B that shall touch the right Line FE.

PRO-

PROBLEM XXXVIII.

To describe a Circle through a given Point that shall touch two right Lines given by Position. [Vide Figure 56.]

N. B. *This Proposition is resolv'd as Prop. 37. for the Point A being given, there is also given the other Point B.*

SUPPOSE the given Point to be A; and let EF, FG, be the two right Lines given by Position, and AEG the Circle sought touching the same, and passing through that Point A. Let the Angle EFG be bisected by the right Line CF, and the Center of the Circle will be found therein. Let that be C; and having let fall the Perpendiculars CE, CG to EF and FG, E and G will be the Points of Contact. Now in the Triangles CEF, CGF, since the Angles E and G are right ones, and the Angles at F are halves of the Angle EFG, all the Angles are given, and consequently the Ratio of the Side CF to CE or CG. Let that be as d to e: and if for determining the Center of the Circle sought C, there be assum'd $CF = x$, CE or CG will be $= \frac{ex}{d}$. Besides, let fall the Perpendicular AH to FC, and since the Point A is given, the right Lines AH and FH will be given. Let them be call'd a and b, and taking FC or x from FH or b, there will remain $CH = b - x$. To whose Square $bb - 2bx + xx$ add the Square of AH or aa, and the Sum $aa + bb - 2bx + xx$ will be ACq by the 47. 1. *Eucl.* because the Angle AHC is, by Supposition, a right one. Now make the Radii of the Circle AC and CG equal to each other; that is, make an Equality between their Values, or between their Squares, and you'll have the Æquation $aa + bb - 2bx + xx = \frac{eexx}{dd}$. Take away xx from both Sides, and changing all the Signs, you'll have $-aa - bb + 2bx = xx - \frac{eexx}{dd}$. Multiply all by dd, and divide by $dd - ee$, and it will become

$-aadd$

$\frac{-aadd-bbdd+2bddx}{dd-ee}=xx$. The Root of which Æquation being extracted, is

$x=\frac{bdd-d\sqrt{eebb+eeaa-ddaa}}{dd-ee}$. Therefore the Length of FC is found, and consequently the Point C, which is the Center of the Circle sought.

If the found Value x, or FC, be taken from b, or HF, there will remain $HC=\frac{-eeb+d\sqrt{eebb+eeaa-ddaa}}{dd-ee}$ the same Æquation which came out in the former Problem, for determining the Length of DC.

PROBLEM XXXIX.

To describe a Circle through two given Points, which shall touch another Circle given by Position. [Vide Problem 11, and Figure 57.]

LET AB, be the two Points given, EK the Circle given by Magnitude and Position, F its Center, ABE the Circle sought, passing through the Points A and B, and touching the other Circle in E, and let C be its Center. Let fall the Perpendiculars CD and FG to AB being produc'd, and draw CF cutting the Circles in the Point of Contact E, and draw also FH parallel to DG, and meeting CD in H. These being [thus] constructed, make AD or $DB=a$, DG or $HF=b$, $GF=c$, and EF (the Radius of the Circle given) $=d$, and $DC=x$; and CH will be ($=CD-FG$) $=x-c$, and CFq ($=CHq+HFq$) $=xx-2cx+cc+bb$, and CBq ($=CDq+DBq$) $=xx+aa$, and consequently CB or $CE=\sqrt{xx+aa}$. To this add EF, and you'll have $CF=d+\sqrt{xx+aa}$, whose Square $dd+aa+xx+2d\sqrt{xx+aa}$, is $=$ to the Value of the same CFq found before, *viz.* $xx-2cx+cc+bb$. Take away from both Sides xx, and there will remain $dd+aa+2d\sqrt{xx+aa}=cc+bb-2cx$. Take away moreover $dd+aa$, and there will come out $2d\sqrt{xx+aa}=cc+bb-dd-aa-2cx$. Now, for Abbreviation sake, for

 $cc+$

$cc + bb - dd - aa$, write $2gg$, and you'll have $2d\sqrt{xx+aa} = 2gg - 2cx$, or $d\sqrt{xx+aa} = gg - cx$. And the Parts of the Æquation being squar'd, there will come out $ddxx + ddaa = g^4 - 2ggcx + ccxx$. Take from both Sides $ddaa$ and $ccxx$, and there will remain $ddxx - ccxx = g^4 - ddaa - 2ggcx$. And the Parts of the Æquation being divided by $dd - cc$, you'll have $xx = \frac{g^4 - ddaa - 2ggcx}{dd - cc}$. And by Extraction of the affected Root $x = \frac{-ggc + \sqrt{g^4dd - d^4aa + ddaacc}}{dd - cc}$.

Having found therefore x, or the Length of DC, bisect AB in D, and at D erect the Perpendicular $DC = \frac{-ggc + d\sqrt{g^4 - aadd + aacc}}{dd - cc}$. Then from the Center C, through the Point A or B, describe the Circle ABE; for that will touch the other Circle EK, and pass through both the Points A, B. Q. E. F.

PROBLEM XL.

To describe a Circle through a given Point which shall touch a given Circle, and also a right Line, both given in Position, [Vide Figure 58.]

LET the Circle to be describ'd be BD, its Center C, and B a Point through which it is to be describ'd, and AD the right Line which it shall touch; the Point of Contact D, and the Circle which it shall touch GEM, its Center F, and its Point of Contact E. Produce CD to Q, so that DQ shall be $= EF$, and through Q draw QN parallel to AD. Lastly, from B and F to AD and QN, let fall the Perpendiculars BA, FN; and from C to AB and FN let fall the Perpendiculars CK, CL. And since $BC = CD$, or AK, BK will be $(= AB - AK) = AB - BC$, and consequently $BKq = ABq - AB \times BC + BCq$. Subtract this from BCq, and there will remain $2AB \times BC - ABq$ for the Square of CK. Therefore $AB \times \overline{2BC - AB} = CKq$; and for the same Reason $FN \times \overline{2FC - FN} = CLq$,

CLq, and consequently $\frac{CKq}{AB}+AB=2BC$, and $\frac{CLq}{FN}+FN=2FC$. Wherefore, if for AB, CK, FN, KL, and CL, you write a, y, b, c, and $c-y$, you'll have $\frac{yy}{2a}+\frac{1}{2}a=BC$, and $\frac{cc-2cy+yy}{2b}+\frac{1}{2}b=FC$. From FC take away BC, and there remains $EF=\frac{cc-2cy+yy}{2b}+\frac{1}{2}b-\frac{yy}{2a}-\frac{1}{2}a$. Now, if the Points where FN being produc'd cuts the right Line AD, and the Circle GEM be mark'd with the Letters H, G, and M, and upon HG produc'd you take $HR=AB$, since HN ($=DQ=EF$) is $=GF$, by adding FH on both Sides, you'll have $FN=GH$, and consequently $AB-FN$ ($=HR-GH$) $=GR$, and $AB-FN+2EF$; that is, $a-b+2EF=RM$, and $\frac{1}{2}a-\frac{1}{2}b+EF=\frac{1}{2}RM$. Wherefore, since above EF was $=\frac{cc-2cy+yy}{2b}+\frac{1}{2}b-\frac{yy}{aa}-\frac{1}{2}a$, if this be written for EF you'll have $\frac{1}{2}RM=\frac{cc-2cy+yy}{2b}-\frac{yy}{2a}$. Call therefore RM d, and d will be $=\frac{cc-2cy+yy}{b}-\frac{yy}{a}$. Multiply all the Terms by a and b, and there will arise $abd=acc-2acy+ayy-byy$. Take away from both Sides $acc-2acy$, and there will remain $abd-acc+2acy=ayy-byy$. Divide by $a-b$, and there will arise $\frac{abd-acc+2acy}{a-b}=yy$. And extracting the Root $y=\frac{ac}{a-b}\pm\sqrt{\frac{aabd-abbd+abcc}{aa-2ab+bb}}$. Which Conclusions may be thus abbreviated; make $c:b::d:e$, then $a-b:a::c:f$; and $fe-fc+2fy$ will be $=yy$, or $y=f\pm\sqrt{ff+fe-fc}$. Having found y, or KC, or AD, take $AD=f\pm\sqrt{ff+fe-fc}$, and at D erect the Perpendicular DC ($=BC$) $=\frac{KCq}{2AB}+\frac{1}{2}AB$; and from the Center C, at the Interval CB or CD, describe the Circle BDE, for this passing

thro[illegible]

through the given Point B, will touch the right Line AD in D, and the Circle GEM in E. Q. E. F.

Hence alſo a Circle may be deſcrib'd which ſhall touch two given Circles, and a right Line given by Poſition. [*Vide Figure* 59.] For let the given Circles be RT, SV, their Centers B F, and the right Line given by Poſition PQ. From the Center F, with the Radius $FS - BR$, deſcribe the Circle EM. From the Point B to the right Line PQ let fall the Perpendicular BP, and having produc'd it to A, ſo that PA ſhall be $= BR$, through A draw AH parallel to PQ, and deſcribe a Circle which ſhall paſs through the Point B, and touch the right Line AH and the Circle EM. Let its Center be C; join BC, cutting the Circle RT in R, and the Circle RS deſcrib'd from the ſame Center C, and the Radius CR will touch the Circles RT, SV, and the right Line PQ, as is manifeſt by the Conſtruction.

PROBLEM XLI.

To deſcribe a Circle that ſhall paſs through a given Point, and touch two other Circles given in Poſition and Magnitude. [Vide Figure 60.]

LET the given Point be A, and let the Circles given in Magnitude and Poſition be TIV, RHS, their Centers C and B; the Circle to be deſcrib'd AIH, its Center D, and the Points of Contact I and H. Join AB, AC, AD, DB, and let AB produc'd cut the Circle RHS in the Points R and S, and AC produc'd, cut the Circle TIV in T and V. And having let fall the Perpendiculars DE from the Point D to AB, and DF from the Point D to AC meeting AB in G, and [alſo the Perpendicular] CK to AB; in the Triangle ADB, $ADq - DBq + ABq$ will be $= 2AE \times AB$, by the 13th of the 2*d. Elem.* But $DB = AD + BR$, and conſequently $DBq = ADq + 2AD \times BR + BRq$. Take away this from $ADq + ABq$, and there will remain $ABq - 2AD \times BR - BRq$ for $2AE \times AB$. Moreover, $ABq - BRq$ is $= \overline{AB - BR} \times \overline{AB + BR} = AR \times AS$. Wherefore, $AR \times AS - 2AD \times BR = 2AE \times AB$. And $\dfrac{AR \times AS - 2AB \times AE}{BR}$

$= 2AD$

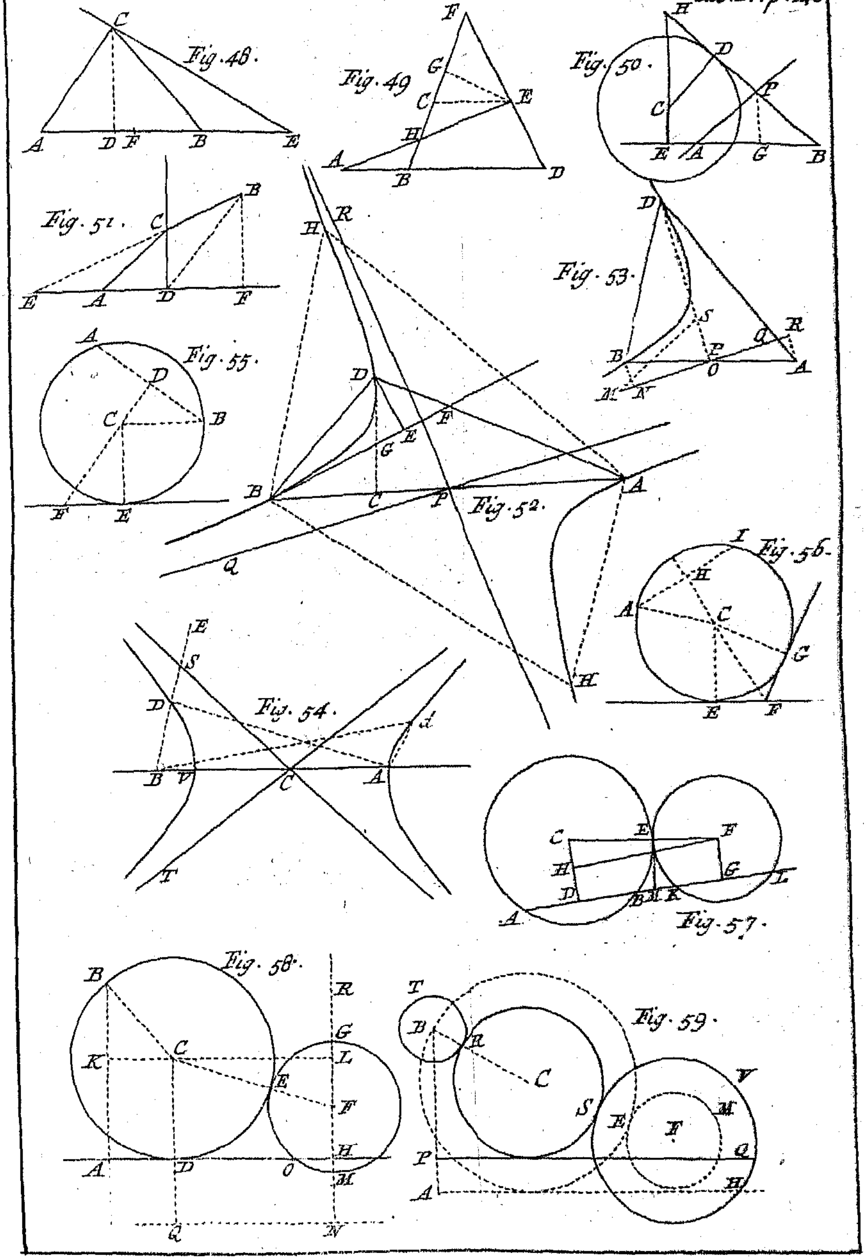
Fig. 48.
Fig. 49
Fig. 50.
Fig. 51.
Fig. 52.
Fig. 53.
Fig. 54.
Fig. 55.
Fig. 56.
Fig. 57.
Fig. 58.
Fig. 59.

$=2AD$. And by a like Reasoning in the Triangle ADC, there will come out again $2AD = \frac{TAV - 2CAF}{CT}$. Wherefore $\frac{RAS - 2BAE}{BR} = \frac{TAV - 2CAF}{CT}$. And $\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR} = \frac{2CAF}{CT}$. And $\overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AC} = AF$. Whence since $AK : AC :: AF : AG$, AG will $= \overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AK}$. Take away this from AE, or $\frac{2KAE}{CT} \times \frac{CT}{2AK}$ and there will remain $GE = \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2AK}$. Whence since $KC : AK :: GE : DE$; DE will be $= \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2KC}$. Upon AB take AP, which let be to AB as CT to BR, and $\frac{2PAE}{CT}$ will be $= \frac{2BAE}{BR}$, and so $\frac{2PK \times AE}{CT} = \frac{2BAE}{BR} - \frac{2KAE}{CT}$, and so $DE = \frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2PK \times AE}{CT} \times \frac{CT}{2KC}$. Upon AB erect the Perpendicular $AQ = \frac{RAS}{BR} - \frac{TAV}{CT} \times \frac{CT}{2KC}$, and in it take $QO = \frac{PK \times AE}{KC}$, and AO will be $= DE$. Join DO, DQ, and CP, and the Triangles DOQ, CKP, will be similar, because their Angles at O and K are right ones, and the Sides ($KC : PK ::$ AE, or $DO : QO$) proportional. Therefore the Angles OQD, KPC, are equal, and consequently QD is perpendicular to CP. Wherefore if AN be drawn parallel to CP, and meeting QD in N, the Angle ANQ will be a right one, and the Triangles AQN, PCK similar; and consequently $PC : KC :: AQ : AN$. Whence since AQ is $\frac{RAS}{BR}$

$\frac{RAS}{BR} - \frac{TAV}{CT} \times \frac{CT}{2KC}$, AN will be $\frac{RAS}{BR} - \frac{TAV}{CT} \times \frac{CT}{2PC}$. Produce AN to M, so that NM shall be $= AN$, and AD will $= DM$, and consequently the Circle will pass through the Point M.

Since therefore the Point M is given, there follows this Resolution of the Problem, without any farther Analysis.

Upon AB take AP, which must be to AB as CT to BR; join CP, and draw parallel to it AM, which shall be to $\frac{RAS}{BR} - \frac{TAV}{CT}$, as CT to PC; and by the Help of the 39th *Probl.* describe through the Points A and M the Circle $AIHM$, which shall touch either of the Circles TIV, RHS, and the same Circle shall touch both. Q. E. F.

And hence also a Circle may be describ'd, which shall touch three Circles given in Magnitude and Position. For let the Radii of the given Circles be A, B, C, and their Centers D, E, F. From the Centers E and F, with the Radii $B \pm A$ and $C \pm A$ describe two Circles, and let a third Circle which touches these [two] be also describ'd, and let it pass through the Point A; let its Radius be G, and its Center H, and a Circle describ'd on the same Center H, with the Radius $G \pm A$, shall touch the three former Circles, as was requir'd.

PRO-

Problem XLII.

Three Staves being erected, or set up an End, in some certain Part of the Earth perpendicular to the Plane of the Horizon, in the Points A, B, and C, whereof that which is in A is six Foot long, that in B eighteen, and that in C eight, the Line AB being thirty Foot long; it happens on a certain Day [in the Year] that the End of the Shadow of the Staff A passes through the Points B and C, and of the Staff B through A and C, and of the Staff C through the Point A. To find the Sun's Declination, and the Elevation of the Pole, or the Day and Place where this shall happen. [Vide Figure 61.]

Because the Shadow of each Staff describes a Conick Section, or the Section of a luminous Cone, whose Vertex is the Top of the Staff; I will feign $BCDEF$ to be such a Curve, [whether it be an Hyperbola, Parabola, or Ellipse] as the Shadow of the Staff A describes that Day, by putting AD, AE, AF, to have been its Shadows, when BC, BA, CA, were respectively the Shadows of the Staves B and C. And besides I will suppose PAQ to be the Meridional Line, or the Axis of this Curve, to which the Perpendiculars BM, CH, DK, EN, and FL, being let fall, are Ordinates. And I will denote these Ordinates indefinitely [or indifferently] by the Letter y, and the intercepted Parts of the Axis AM, AH, AK, AN, and AL by the Letter x. I'll suppose, lastly, the Æquation $aa \perp bx \perp cxx = yy$, to express the Relation of x and y, (*i. e.* the Nature of the Curve) assuming aa, b, and c, as known Quantities, as they will be found to be from the Analysis. Where I made the unknown Quantities of two Dimensions only because the Æquation is [to express] a Conick Section: and I omitted the odd Dimensions of y, because it is an Ordinate to the Axis. And I denoted the Signs of b and c, as being indeterminate by the Note $\perp$, which I use indifferently

rently for $+$ or $-$, and its opposite $\top$ for the contrary. But I made the Sign of the Square aa Affirmative, because the concave Part of the Curve necessarily contains the Staff A, projecting its Shadows to the opposite Parts (C and F; D and E); and then, if at the Point A you erect the Perpendicular $A\beta$, this will some where meet the Curve, suppose in β, that is, the Ordinate y, where x is nothing, will [still] be real. From thence it follows that its Square, which in that Case is aa, will be Affirmative.

It is manifest therefore, that this fictitious Æquation $aa \perp bx \perp cxx = yy$, as it is not fill'd with superfluous Terms, so neither is it more restrain'd [or narrower] than what is capable of satisfying all the Conditions of the Problem, and will denote the Hyperbola, Ellipse, or Parabola, according as the Values of aa, b, c, shall be determin'd, or found to be nothing but what may be their Value; and with what Signs b and c are to be affected, and thence what Sort of a Curve this may be, will be manifest from the following Analysis.

The former Part of the Analysis.

Since the Shadows are as the Altitude of the Staves, you'll have $BC : AD :: AB : AE$ $(:: 18 : 6) :: 3 : 1$. Also $CA : AF$ $(:: 8 : 6) :: 4 : 3$. Wherefore naming (or making] $AM = +r$, $MB = \perp s$, $AH = \perp t$, and $HC = +v$. From the Similitude of the Triangles AMB, ANE, and AHC, ALF, AN will be $= -\frac{r}{3}$. $NE = \top\frac{s}{3}$. $AL = \top\frac{3t}{4}$, and $LF = -\frac{3v}{4}$; whose Signs I put contrary to the Signs of AM, MB, AH, HC, because they tend contrary Ways with respect to the Point A from which they are drawn, and from the Axis PQ on which they stand. Now these being respectively written for x and y in the fictitious Æquation $aa \perp bx \perp cxx = yy$.

r and $\perp s$ will give $aa \perp br \perp crr = ss$.
$-\frac{r}{3}$ and $\top\frac{s}{3}$ will give $aa \top \frac{br}{3} \perp \frac{1}{9}crr = \frac{1}{9}ss$.
$\perp t$ and $+v$ will give $aa \perp b \times \perp t \perp ctt = vv$.
$\top\frac{3}{4}t$ and $-\frac{3}{4}v$ will give $aa \perp \frac{3}{4}b \times \top t \perp \frac{9}{16}ctt = \frac{9}{16}vv$.

Now

Now, by exterminating ss from the first and second Æquations, in order to obtain r, there comes out $\frac{2aa}{\perp b}=r$. Whence it is manifest, that $\perp b$ is Affirmative, because r is so. Also by exterminating vv from the third and fourth, (after having written for $\perp b$ its Value $+b$) to obtain t, there comes out $\frac{aa}{3b}=\perp t$, therefore t is positive and equal to $\frac{aa}{3b}$, and having writ $\frac{2aa}{b}$ for r in the first, and $\frac{aa}{3b}$ for t in the third, there arise $3aa\perp\frac{4a^4c}{bb}=ss$, and $\frac{4}{3}aa\perp\frac{a^4c}{9bb}=vv$.

Moreover, having let fall $B\lambda$ perpendicular upon CH, BC will be : AD (: : 3 : 1) : : $B\lambda$: AK : : $C\lambda$: DK. Wherefore, since $B\lambda$ is $(=AM-AH=r-t)=\frac{5aa}{3b}$, AK will be $=\frac{5aa}{9b}$, but with a Negative Sign, *viz.* $-\frac{5aa}{9b}$. Also since $C\lambda$ $(=CH\perp BM=v\perp s)=\sqrt{\frac{4aa}{3}\perp\frac{a^4c}{9bb}}\perp\sqrt{3aa\perp\frac{4a^4c}{bb}}$, and therefore $DK\ (=\frac{1}{3}C\lambda)=\sqrt{\frac{4aa}{27}\perp\frac{a^4c}{81bb}}\perp\sqrt{\frac{1}{3}aa\perp\frac{4a^4c}{9bb}}$; which being respectively written in the Æquation $aa\perp bx\perp cxx=yy$, or rather the Æquation $aa+bx\perp cxx=yy$, because b hath before been found to be Positive, for AK and DK, or x and y, there comes out $\frac{4aa}{9}\perp\frac{25a^4c}{81bb}=\frac{13}{27}aa\perp\frac{37a^4c}{81bb}\perp 2\sqrt{\frac{4aa}{27}\perp\frac{a^4c}{81bb}}\times\sqrt{\frac{aa}{3}\perp\frac{4a^4c}{9bb}}$. And by Reduction $-bb\top 4aac=\perp 2\sqrt{36b^4\perp 51aabbc+4a^4cc}$; and the Parts being squar'd, and again reduc'd, there comes out $0=143b^4\perp 196aabbc$, or $\frac{-143bb}{196aa}=\perp c$. Whence it is manifest, that $\perp c$ is Negative, and consequently the fictitious Æquation $aa \perp bx \perp cxx = yy$ will be of this Form, $aa + bx - cxx = yy$. And its Center and two Axes are thus found.

Making $y = 0$, as happens in the Vertex's of the Figure P and Q, you'll have $aa + bx = cxx$, and having extracted the Root $x = \frac{b}{2c} \pm \sqrt{\frac{bb}{4cc} + \frac{aa}{c}} = \left\{ \begin{matrix} AQ \\ AP \end{matrix} \right\}$, that is, $AQ = \frac{b}{2c} + \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$, and $AP = \frac{b}{2c} - \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$, where AP and AQ are computed from A towards the Parts Q; and consequently when AP is computed from A towards P, its Value will be found to be $-\frac{b}{2c} + \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$. And consequently, taking $AV = \frac{b}{2c}$, V will be the Center of the Ellipse, and VQ, or VP, $\left(\sqrt{\frac{bb}{4cc} + \frac{aa}{c}}\right)$ the greatest Semi-Axis. If, moreover, the Value of AV, or $\frac{b}{2c}$, be put for x in the Æquation $aa + bx - cxx = yy$, there will come out $aa + \frac{bb}{4c} = yy$. Wherefore $aa + \frac{bb}{4c}$ will be $= VZq$, that is, to the Square of the least Semi-Axis. Lastly, in the Values of AV and VQ, VZ already found, writing $\frac{143bb}{196aa}$ for c, there come out $\frac{98aa}{143b} = AV$, $\frac{112aa\sqrt{3}}{143b} = VQ$, and $\frac{8a\sqrt{3}}{\sqrt{143}} = VZ$.

The other Part of the Analysis. [Vide Figure 62.]

Suppose now the Staff AR standing on the Point A, and RPQ will be the Meridional Plane, and $RPZQ$ the luminous Cone whose Vertex is R. Let moreover TXZ be a Plane cutting the Horizon in VZ, and the Meridional Plane in TVX, which Section let it be perpendicular to the

the Axis of the World, or of the Cone, and it will cut the Cone in the Periphery of the Circle TZX, which will be every where at an equal Diſtance, as RX, RZ, RT, from its Vertex. Wherefore, if PS be drawn parallel to TX, you'll have $RS = RP$, by reaſon of the equal Quantities RX, RT; and alſo $SX = XQ$, by reaſon of the equal Quantities PV, VQ; whence RX or RZ $\left(=\frac{RS+RQ}{2}\right)$ $=\frac{RP+RQ}{2}$. Laſtly, draw RV, and ſince VZ perpendicularly ſtands on the Plane RPQ, (as being the Section of the Planes perpendicularly ſtanding on the ſame [Plane]) the Triangle RVZ will be right-angled at V.

Now making $RA = d$, $AV = e$, VP or $VQ = f$, and $VZ = g$, you'll have $AP = f - e$, and $RP = \sqrt{ff - 2ef + ee + dd}$. Alſo $AQ = f + e$, and $RQ = \sqrt{ff + 2ef + ee + dd}$; and conſequently RZ $\left(=\frac{RP+RQ}{2}\right) = \frac{\sqrt{ff-2ef+ee+dd} + \sqrt{ff+2ef+ee+dd}}{2}$. Whoſe Square $\frac{dd+ee+ff}{2} + \frac{1}{2}$. $\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4}$, is equal $(RVq + VZq = RAq + AVq + VZq)$ = to $dd + ee + gg$. Now having reduc'd $\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4} = dd + ee - ff + 2gg$, and the Parts being ſquar'd and reduc'd into Order, $ddff = ddgg + eegg - ffgg + g^4$, or $\frac{ddff}{gg} = dd + ee - ff + gg$. Laſtly, 6, $\frac{98aa}{143b}$, $\frac{112aa\sqrt{3}}{143b}$, $\frac{8a\sqrt{3}}{\sqrt{143}}$ (the Values of AR, AV, VQ, and VZ) being reſtor'd for d, e, f, and g, there ariſes $36 - \frac{196a^4}{143bb} + \frac{192aa}{143} = \frac{36, 14, 14aa}{143bb}$, and thence by Reduction $\frac{49a^4 + 36{,}49aa}{48aa + 1287} = bb$.

In the firſt Scheme $AMq + MBq = ABq$, that is, $rr + ss = 33 \times 33$. But r was $= \frac{2aa}{b}$, and $ss = 3aa -$

$\frac{4a^4c}{bb}$, whence $rr = \frac{4a^4}{bb}$, and substituting $\frac{143bb}{196aa}$ for c) $ss = \frac{4aa}{49}$. Wherefore $\frac{4a^4}{bb} + \frac{4aa}{49} = 33 \times 33$, and thence by Reduction there again results $\frac{4{,}49a^4}{53361 - 4aa} = bb$. Putting therefore an Equality between the two Values of bb, and dividing each Part of the Æquation by 49, you'll have $\frac{a^4 + 36aa}{48aa + 1287} = \frac{4a^4}{53361 - 4aa}$; whose Parts being multiply'd cross-ways, and divided by 49, there comes out $4a^4 = 981aa + 274428$, whose Root aa is $\frac{981 + \sqrt{1589625}}{8} = 280{,}2254144$.

Above was found $\frac{4{,}49a^4}{53361 - 4aa} = bb$, or $\frac{14aa}{\sqrt{53361 - 4aa}} = b$. Whence AV $(\frac{98aa}{143b})$ is $\frac{7\sqrt{53361 - 4aa}}{143}$, and VP, or VQ $(\frac{112aa\sqrt{3}}{143b})$ is $\frac{8}{143}\sqrt{160083 - 12aa}$. That is, by substituting 280,2254144 for aa, and reducing the Terms into Decimals, $AV = 11{,}188297$, and VP or $VQ = 22{,}147085$; and consequently AP $(PV - AV) = 10{,}958788$, and AQ $(AV + VQ)$ 33,335382.

Lastly, if $\frac{1}{7}AR$ or 1 be made Radius, $\frac{1}{7}AQ$ or 5,355897 will be the Tangent of the Angle ARQ of 79 *gr.* 47'. 48''. and $\frac{1}{7}AP$ or 1,826465 the Tangent of the Angle ARP of 61 *gr.* 17'. 52''. half the Sum of which Angles 70 *gr.* 32'. 50''. is the Complement of the Sun's Declination; and the Semi-difference 9 *gr.* 14'. 58''. the Complement of the Latitude of the Place. Then, the Sun's Declination was 19 *gr.* 27'. 10''. and the Latitude of the Place 80 *gr.* 45'. 20''. which were to be found.

PROBLEM XLIII.

If at the Ends of the Thread DAE, moving upon the fix'd Tack A, there are hang'd two Weights, D and E, whereof the Weight E slides through the oblique Line BG given in Position; to find the Place of the Weight E, where these Weights are in Æquilibrio. [Vide Figure 63.]

SUPPOSE the Problem done, and parallel to AD draw EF, which shall be to AE as the Weight E to the Weight D. And from the Points A and F to the Line BG let fall the Perpendiculars AB, FG. Now since the Weights are, by Supposition, as the Lines AE and EF, express those Weights by those Lines, the Weight D by the Line EA, and the Weight E by the Line EF. Therefore the Body, or E, directed by the Force of its own Weight, tends towards F. And by the oblique Force EG tends towards G. And the same Body E by the Weight D in the direct Force AE, is drawn towards A, and in the oblique Force BE is drawn towards B. Since therefore the Weights sustain each other in Æquilibrio, the Force by which the Weight E is drawn towards B, ought to be equal to the contrary Force by which it tends towards G, that is, BE ought to be equal to EG. But now the Ratio of AE to EF is given by the Hypothesis; and by reason of the given Angle FEG, there is also given the Ratio of FE to EG, to which BE is equal. Therefore there is given the Ratio of AE to BE. AB is also given in Length; and thence the Triangle ABE, and the Point E will easily be given. *Viz.* make $AB = a$, $BE = x$, and AE will be equal $\sqrt{aa + xx}$; moreover, let AE be to BE in the given Ratio of d to e, and $e\sqrt{aa + xx}$ will $= dx$. And the Parts of the Æquation being squar'd and reduc'd, $eeaa = ddxx - eexx$, or $\frac{ea}{\sqrt{dd - ee}} = x$. Therefore the Length BE is found, which determines the Place of the Weight E. Q. E. F.

Now,

Now, if both Weights descend by oblique Lines given in Position, the Computation may be made thus. [*Vide Figure* 64.] Let CD and BE be oblique Lines given by Position, through which those Weights descend. From the fix'd Tack A to these Lines let fall the Perpendiculars AC, AB, and let the Lines EG, DH, erected from the Weights perpendicularly to the Horizon, meet them in the Points G and H; and the Force by which the Weight E endeavours to descend in a perpendicular Line, or the whole Gravity of E, will be to the Force by which the same Weight endeavours to descend in the oblique Line BE, as GE to BE; and the Force by which it endeavours to descend in the oblique Line BE, will be to the Force by which it endeavours to descend in the Line AE, that is, to the Force by which the Thread AE is distended [or stretch'd] as BE to AE. And consequently the Gravity of E will be to the Tension of the Thread AE, as GE to AE. And by the same Ratio the Gravity of D will be to the Tension of the Thread AD, as HD to AD. Let therefore the Length of the whole Thread $DA + AE$ be c, and let its Part $AE = x$, and its other Part AD will $= c - x$. And because $AEq - ABq$ is $= BEq$, and $ADq - ACq = CDq$; let, moreover, $AB = a$, and $AC = b$, and BE will be $= \sqrt{xx - aa}$, and $CD = \sqrt{xx - 2cx + cc - bb}$. Moreover, since the Triangles BEG, CDH are given in Specie, let $BE : EG :: f : E$, and $CD : DH :: f : g$, and EG will $= \frac{E}{f}\sqrt{xx - aa}$, and $DH = \frac{g}{f}\sqrt{xx - 2cx + cc - bb}$. Wherefore since $GE : AE ::$ Weight E : Tension of AE; and $HD : AD ::$ Weight D : Tension of AD; and those Tensions are equal, you'll have $\dfrac{Ex}{\frac{E}{f}\sqrt{xx - aa}} =$ Tension of $AE =$ to the Tension $AD = \dfrac{Dc - Dx}{\frac{g}{f}\sqrt{xx - 2cx + cc - bb}}$; from the Reduction of which Æquation there comes out $gx\sqrt{xx - 2cx + cc - bb} = \overline{Dc - Dx}\sqrt{xx - aa}$, or

$$\left.\begin{matrix} gg \\ -DD \end{matrix}\right. x^4 \begin{matrix} -2ggc \\ +2DDc \end{matrix}\, x^3 \begin{matrix} +ggcc \\ -ggbb \\ -DDcc \\ +DDaa \end{matrix}\, xx - 2DDcaax + DDccaa = 0.$$

But

But if you desire a Case wherein this Problem may be constructed by a Rule and Compass, make the Weight D to the Weight E as the Ratio $\frac{BE}{EG}$ to the Ratio $\frac{CD}{DH}$, and g will become $= D$; and so in the Room of the precedent Æquation you'll have this, $-\frac{aa}{bb}xx - 2acx + aacc = 0$, or $x = \frac{ac}{a+b}$.

Problem XLIV.

If on the String DABCF, that slides about the two Tacks A and B, there are hung three Weights, D, E, F; D and F at the Ends of the String, and E at its middle Point C, plac'd between the Tacks: From the given Weights and Position of the Tacks to find the Situation of the Point C, where the middle Weight hangs, and where they are in Æquilibrio. [Vide Figure 65.]

SINCE the Tension of the Thread AC is equal to the Tension of the Thread AD, and the Tension of the Thread BC to the Tension of the Thread BF, the Tension of the Strings or Threads AC, BC, EC will be as the Weights D, E, F. Then take the Parts of the Thread CG, CH, CI, in the same Ratio as the Weights. Compleat the Triangle GHI. Produce IC till it meet GH in K, and GK will be $= KH$, and $CK = \frac{1}{2}CI$, and consequently C the Center of Gravity of the Triangle GHI. For, draw PQ through C, perpendicular to CE, and perpendicular to that, from the Points G and H, draw GP, HQ. And if the Force by which the Thread AC by the Weight D draws the Point C towards A, be express'd by the Line GC, the Force by which that Thread will draw the same Point towards P, will be express'd by the Line CP; and the Force by which it draws it to K, will be express'd by the Line GP. And in like Manner, the Forces by which the Thread BC, by Means of the Weight F, draws the same Point C towards

towards B, Q, and K, will be expreſs'd by the Lines CH, CK, and HQ; and the Force by which the Thread CE, by Means of the Weight E, draws that Point C towards E, will be expreſs'd by the Line CI. Now ſince the Point C is ſuſtain'd in Æquilibrio by equal Forces, the Sum of the Forces by which the Threads AC and BC do together draw C towards K, will be equal to the contrary Force by which the Thread EC draws that Point towards E; that is, the Sum $GP + HQ$ will be equal to CI; and the Force by which the Thread AC draws the Point C towards P, will be equal to the contrary Force by which the Thread BC draws the ſame Point C towards Q; that is, the Line PC is equal to the Line CQ. Wherefore, ſince PG, CK, and QH are Parallel, GK will be alſo $= KH$, and CK ($= \frac{GP + HQ}{2} = \frac{1}{2}CI$.) Which was to be ſhewn. It remains therefore to determine the Triangle GCK, whoſe Sides GC and HC are given, together with the Line CK, which is drawn from the Vertex C to the middle of the Baſe. Let fall therefore from the Vertex C to the Baſe CH the Perpendicular CL, and $\frac{GCq - CHq}{2GH}$ will be $= KL = \frac{GCq - KCq - GKq}{2GK}$. For $2GK$ write GH, and having rejected the common Diviſor GH, and order'd the Terms, you'll have $GCq - 2KCq + CHq = 2GKq$, or $\sqrt{\frac{1}{2}GCq - KCq + \frac{1}{2}CHq} = GK$, having found GK, or KH, there are given together the Angles GCK, KCH, or DAC, FBC. Wherefore, from the Points A and C in theſe given Angles DAC, FBC, draw the Lines AC, BC, meeting in the Point C; and C will be the Point ſought.

But it is not always neceſſary to ſolve Queſtions that are of the ſame Kind, particularly by Algebra, but from the Solution of one of them you may moſt commonly infer the Solution of the other. As if now there ſhould be propos'd this Queſtion.

The

The Thread ACDB being divided into the given Parts AC, CD, DB, and its Ends being fasten'd to the two Tacks given by Position, A and B; and if at the Points of Division, C and D, there are hang'd the two Weights E and F; from the given Weight F, and the Situation of the Points C and D, to know the Weight E. [Vide Figure 66.]

FROM the Solution of the former Problem the Solution of this may be easily enough found. Produce the Lines AC, BD, till they meet the Lines DF, CE in G and H; and the Weight E will be to the Weight F, as DG to CH.

And hence may appear a Method of making a Balance of only Threads, by which the Weight of any Body E may be known, from only one given Weight F.

PROBLEM XLV.

A Stone falling down into a Well, from the Sound of the Stone striking the Bottom, to determine the Depth of the Well.

LET the Depth of the Well be x, and if the Stone descends with an uniformly accelerated Motion through any given Space a, in any given Time b, and the Sound passes with an uniform Motion through the same given Space a, in the given Time d, the Stone will descend through the Space x in the Time $b\sqrt{\frac{x}{a}}$; but the Sound which is caus'd by the Stone striking upon the Bottom of the Well, will ascend by the same Space x, in the Time $\frac{dx}{a}$. For the Spaces describ'd by descending heavy Bodies, are as the Squares of the Times of Descent; or the Roots of the Spaces, that is, $\sqrt{x}$ and $\sqrt{a}$ are as the Times themselves. And the Spaces x and a, through which the Sound passes, are as the Times of Passage. And the Sum of these Times

 $b\sqrt{x}$

$b\sqrt{\frac{x}{a}}$, and $\frac{dx}{a}$, is the Time of the Stone's falling to the Return of the Sound. This Time may be known by Observation. Let it be t, and you'll have $b\sqrt{\frac{x}{a}} + \frac{dx}{a} = t$. And $b\sqrt{\frac{x}{a}} = t - \frac{dx}{a}$. And the Parts being squar'd, $\frac{bbx}{a} = tt - \frac{2tdx}{a} + \frac{ddxx}{aa}$. And by Reduction $xx = \frac{2adt + abb}{dd}x - \frac{aatt}{dd}$. And having extracted the Root $x = \frac{adt + \frac{1}{2}abb}{dd} - \frac{ab}{2dd}\sqrt{bb + 4dt}$.

PROBLEM XLVI.

Having given the Perimeter and Perpendicular of a right-angled Triangle, to find the Triangle. [Vide Figure 67.]

LET C be the right Angle of the Triangle, ABC and CD a Perpendicular let fall thence to the Base AB. Let there be given $AB + BC + AC = a$, and $CD = b$. Make the Base $AB = x$, and the Sum of the Sides will be $a - x$. Put y for the Difference of the Legs, and the greater Leg AC will be $= \frac{a - x + y}{2}$; the less $BC = \frac{a - x - y}{2}$. Now, from the Nature of a right-angled Triangle you have $ACq + BCq = ABq$, that is, $\frac{aa - 2ax + xx + yy}{2} = xx$. And also $AB : AC :: BC : DC$, therefore $AB \times DC = AC \times BC$, that is, $bx = \frac{aa - 2ax + xx - yy}{4}$. By the former Æquation yy is $= xx + 2ax - aa$. By the latter yy is $= xx - 2ax + aa - 4bx$. And consequently $xx + 2ax - aa = xx - 2ax + aa - 4bx$. And by Reduction $4ax + 4bx = 2aa$, or $x = \frac{aa}{2a + 2b}$.

Geome-

Geometrically thus. In every right-angled Triangle, as the Sum of the Perimeter and Perpendicular is to the Perimeter, so is half the Perimeter to the Base.

Subtract $2x$ from a, and there will remain $\frac{ab}{a+b}$, the Excess of the Sides above the Base. Whence again, as in every right-angled Triangle the Sum of the Perimeter and Perpendicular is to the Perimeter, so is the Perpendicular to the Excess of the Sides above the Base.

PROBLEM XLVII.

Having given the Base AB of a right-angled Triangle, and the Sum of the Perpendicular, and the Legs $CA+CB+CD$; to find the Triangle.

LET $CA+CB+CD=a$, $AB=b$, $CD=x$, and $AC+CB$ will be $=a-x$. Put $AC-CB=y$, and AC will be $=\frac{a-x+y}{2}$, and $CB=\frac{a-x-y}{2}$. But $ACq+CBq$ is $=ABq$; that is, $\frac{aa-2ax+xx+yy}{2}=bb$.

Moreover, $AC\times CB=AB\times CD$, that is, $\frac{aa-2ax+xx-yy}{4}=bx$. Which being compar'd, you have $2bb-aa+2ax-xx=yy=aa-2ax+xx-4bx$. And by Reduction, $xx=2ax+2bx-aa+bb$, and $x=a+b-\sqrt{2ab+2bb}$.

Geometrically thus. In any right-angled Triangle, from the Sum of the Legs and Perpendicular subtract the mean Proportional between the said Sum and the double of the Base, and there will remain the Perpendicular.

The same otherwise.

Make $CA+CB+CD=a$, $AB=b$, and $AC=x$, and BC will be $=\sqrt{bb-xx}$, $CD=\frac{x\sqrt{bb-xx}}{b}$. And $x+CB+CD=a$, or $CB+CD=a-x$. And therefore

$b+x$

$\frac{b+x}{b}\sqrt{bb-xx}=a-x$. And the Parts being squar'd and multiply'd by bb, there will be made $-x^4-2bx^3+2b^3x+b^4=aabb-2abbx+bbxx$. Which Æquation being order'd, by Transposition of Parts, after this Manner, $x^4+2bx^3\left\{\begin{matrix}+3bb\\+2ab\end{matrix}\right\}xx\begin{matrix}+2b^3\\+2abb\end{matrix}x\left\{\begin{matrix}+b^4\\+2ab^3\\+aabb\end{matrix}\right\}$ $=\begin{matrix}2bb\\2ab\end{matrix}xx\left.\begin{matrix}+4b^3\\+4abb\end{matrix}x\right\}\left\{\begin{matrix}+2b^4\\+2ab^3\end{matrix}\right\}$ and extracting the Roots on both Sides, there will arise $xx+bx+bb+ab=\overline{x+b}\sqrt{2ab+2bb}$. And the Root being again extracted $x=-\frac{1}{2}b+\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}\pm\sqrt{b\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}-\frac{1}{4}bb-\frac{1}{2}ab}$.

The Geometrical Construction. [Vide Figure 53.]

Take therefore $AB=\frac{1}{2}b$, $BC=\frac{1}{2}a$, $CD=\frac{1}{2}AB$, AE, a mean Proportional between b and AC, and $EF=Ef$, a mean Proportional between b and DE, and BF, Bf will be the two Legs of the Triangle.

PROBLEM XLVIII.

Having given in the right-angled Triangle ABC, the Sum of the Sides $AC+BC$, and the Perpendicular CD, to find the Triangle.

LET $AC+BC=a$, $CD=b$, $AC=x$, and BC will $=a-x$, $AB=\sqrt{aa-2ax+2xx}$. Moreover, $CD:AC::BC:AB$. Therefore again $AB=\frac{ax-xx}{b}$. Wherefore $ax-xx=b\sqrt{aa-2ax+2xx}$; and the Parts being squar'd and order'd $x^4-2ax^3\begin{matrix}+aa\\-2bb\end{matrix}xx+2abbx-aabb=0$. Add to both Parts $aabb+b^4$, and there will be made $x^4-2ax^3\begin{matrix}+aa\\-2bb\end{matrix}xx+2abbx+b^4=aabb+b^4$. And the Root being extracted on both Sides,

Sides, $xx - ax - bb = -b\sqrt{aa+bb}$, and the Root being again extracted $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb - b\sqrt{aa+bb}}$.

The Geometrical Construction. [Vide Figure 69.]

Take $AB = BC = \frac{1}{2}a$. At C erect the Perpendicular $CD = b$. Produce DC to E, so that DE shall $= DA$. And between CD and CE take a mean Proportional CF. And let a Circle, describ'd from the Center F and the Radius BC, cut the right Line BC in G and H, and BG and BH will be the two Sides of the Triangle.

The same otherwise.

Let $AC + BC = a$, $AC - BC = y$, $AB = x$, and $DC = b$, and $\frac{a+y}{2}$ will $= AC$, $\frac{a-y}{2} = BC$, $\frac{aa+yy}{2} = ACq + BCq = ABq = xx$. $\frac{aa-yy}{4b} = \frac{AC \times BC}{DC} = AB = x$. Therefore $2xx - aa = yy = aa - 4bx$, and $xx = aa - 2bx$, and the Root being extracted $x = -b + \sqrt{bb+aa}$. Whence in the Construction above CE is the Hypothenuse of the Triangle sought. But the Base and Perpendicular, as well in this as the Problem above being given, the Triangle is thus expeditiously constructed. [*Vide Figure* 70.] Make a Parallelogram CG, whose Side CE shall be the Basis of the Triangle, and the other Side CF the Perpendicular. And upon CE describe a Semicircle, cutting the opposite Side PG in H. Draw CH, EH, and CHE will be the Triangle sought.

Problem XLIX.

In a right-angled Triangle, having given the Sum of the Legs, and the Sum of the Perpendicular and Base, to find the Triangle.

LET the Sum of the Legs AC and BC be [call'd] a, the Sum of the Base AB and of the Perpendicular CD be [call'd] b, let the Leg $AC = x$, the Base $AB = y$, and BC will $= a - x$, $CD = b - y$, $aa - 2ax + 2xx = ACq + BCq$

$+ BCq = ABq = yy$, $ax - xx = AC \times BC = AB \times CD = by - yy = by - aa + 2ax - 2xx$, and $by = aa - ax + xx$. Make its Square $a^4 - 2a^3x + 3aaxx - 2ax^3 + x^4$ equal to $yy \times bb$, that is, equal to $aabb - 2abbx + 2bbxx$. And ordering the Æquation, there will come out $x^4 - 2ax^3 \begin{matrix} +3aa \\ -2bb \end{matrix} xx \begin{matrix} -2a^3 \\ +2abb \end{matrix} x \begin{matrix} +a^4 \\ -aabb \end{matrix} = 0$. Add to each Side of the Æquation $b^4 - aabb$, and there will come out $x^4 - 2ax^3 \begin{matrix} +3aa \\ -2bb \end{matrix} xx \begin{matrix} -2a^3 \\ +2abb \end{matrix} x \begin{matrix} +a^4 \\ -2aabb \\ +b^4 \end{matrix} = b^4 - aabb$. And the Root being extracted on both Sides $xx - ax + aa - bb = -b\sqrt{bb - aa}$, and the Root being again extracted $x = \frac{1}{2}a \pm \sqrt{bb - \frac{3}{4}aa - b\sqrt{bb - aa}}$.

The Geometrical Construction.

Take R a mean Proportional between $b + a$ and $b - a$, and S a mean Proportional between R and $b - R$, and T a mean Proportional between $\frac{1}{2}a + S$ and $\frac{1}{2}a - S$; and $\frac{1}{2}a + T$, and $\frac{1}{2}a - T$ will be the Sides of the Triangle.

PROBLEM L.

To subtend the given Angle CBD with the given right Line CD, so that if AD be drawn from the End of that right Line D to the Point A, given on the right Line CB produc'd, the Angle ADC shall be equal to the Angle ABD. [Vide Figure 71.]

MAKE $CD = a$, $AB = b$, $BD = x$, and BD will be $: BA :: CD : DA = \frac{ab}{x}$. Let fall the Perpendicular DE, and BE will be $= \frac{BDq - ADq + BAq}{2BA} = \frac{xx - \frac{aabb}{xx} + bb}{2b}$. By reason of the given Triangle DBA,

make

make $BD : BE :: b : e$, and you'll have again $BE = \frac{ex}{b}$, therefore $xx - \frac{aabb}{xx} + bb = 2ex$. And $x^4 - 2ex^3 + bbxx - aabb = 0$.

Problem LI.

Having the Sides of a Triangle given, to find the Angles. [Vide Figure 72.]

LET the [given] Sides $AB = a$, $AC = b$, $BC = c$, to find the Angle A. Having let fall to AB the Perpendicular CD, which is opposite to that Angle, you'll have in the first Place, $bb - cc = ACq - BCq = ADq - BDq = AD + BD \times AD - BD = AB \times 2AD - AB = 2AD \times a - aa$. And consequently $\frac{1}{2}a + \frac{bb - cc}{2a} = AD$. Whence comes out this first *Theorem*. As AB to $AC + BC$ so $AB - BC$ to a fourth Proportional N. $\frac{AB + N}{2} = AD$. As AC to AD so Radius to the Cosine of the Angle A.

Moreover, $DCq = ACq - ADq = \frac{2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4}{4aa} = \frac{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{-a+b+c}}{4aa}$. Whence having multiply'd the Roots of the Numerator and Denominator by b, there is made this second *Theorem*. As $2ab$ to a mean Proportional between $\overline{a+b+c} \times \overline{a+b-c}$ and $\overline{a-b+c} \times \overline{-a+b+c}$; so is Radius to the Sine of the Angle a.

Moreover, on AB take $AE = AC$, and draw CE, and the Angle ECD will be equal to half the Angle A. Take AD from AE, and there will remain $DE = b - \frac{1}{2}a - \frac{bb - cc}{2a} = \frac{cc - aa + 2ab - bb}{2a} = \frac{\overline{c+a-b} \times \overline{c-a+b}}{2a}$. Whence $DEq = \frac{\overline{c+a-b} \times \overline{c+a-b} \times \overline{c-a+b} \times \overline{c-a+b}}{4aa}$. And hence is made the third and fourth *Theorem*, viz. As

2 a b

$2ab$ to $\overline{c+a-b}\times\overline{c-a+b}$ (so AC to DE) so Radius to the versed Sine of the Angle A. And, as a mean Proportional between $a+b+c$, and $a+b-c$ to a mean Proportional between $c+a-b$, and $c-a+b$ (so CD to DE) so Radius to the Tangent of half the Angle A, or the Cotangent of half the Angle to Radius.

Besides, CEq is $= CDq + DEq =$

$$\frac{2abb+bcc-baa-b^3}{a} = \frac{b}{a}\times\overline{c+a-b}\times\overline{c-a+b}.$$

Whence the fifth and sixth *Theorem*. As a mean Proportional between $2a$ and $2b$ to a mean Proportional between $c+a-b$, and $c-a+b$, or as 1 to a mean Proportional between $\frac{c+a-b}{2a}$, and $\frac{c-a+b}{2b}$ (so AC to $\frac{1}{2}CE$, or CE to DE) so Radius to the Sine of $\frac{1}{2}$ the Angle A. And as a mean Proportional between $2a$ and $2b$ to a mean Proportional between $a+b+c$ and $a+b-c$ (so CE to CD) so Radius to the Cosine of half the Angle A.

But if besides the Angles, the Area of the Triangle be also sought, multiply CDq by $\frac{1}{4}ABq$, and the Root, *viz.* $\frac{1}{4}\sqrt{\overline{a+b+c}\times\overline{a+b-c}\times\overline{a-b+c}\times\overline{-a+b+c}}$ will be the Area sought.

Problem LII.

From the Observation of four Places of a Comet, moving with an uniform right-lined Motion through the Heaven, to determine its Distance from the Earth, and Direction and Velocity of its Motion, according to the Copernican *Hypothesis.* [Vide Figure 73.]

If from the Center of the Comet in the four Places observ'd, you let fall so many Perpendiculars to the Plane of the Ecliptick; and A, B, C, D, be the Points in that Plane on which the Perpendiculars fall; through those Points draw the right Line AD, and this will be cut by the Perpendiculars in the same Ratio with the Line which the Comet describes by its Motion; that is, so that AB shall be to AC as the Time between the first and second Observation to the Time between the first and third; and AB to AD as the Time

Time between the first and second to the Time between the first and fourth. From the Observations therefore there are given the Proportions of the Lines AB, AC, AD, to one another.

Moreover, let the Sun S be in the same Plane of the Ecliptick, and EH an Arch of the Ecliptical Line in which the Earth moves; E, F, G, H, four Places of the Earth in the Times of the Observations, E the first Place, F the second, G the third, H the fourth. Join AE, BF, CG, DH, and let them be produc'd till the three former cut the latter in I, K, and L, *viz.* BF in I, CG in K, DH in L. And the Angles AIB, AKC, ALD will be the Differences of the observ'd Longitudes of the Comet; AIB the Difference of the Longitudes of the first and second Place of the Comet; AKC the Difference of the Longitudes of the first and third Place, and ALD the Difference of the Longitudes of the first and fourth Place. There are given therefore from the Observations the Angles AIB, AKC, ALD.

Join SE, SF, EF; and by reason of the given Points S, E, F, and the given Angle ESF, there will be given the Angle SEF. There is given also the Angle SEA, as being the Difference of Longitude of the Comet and Sun in the Time of the first Observation. Wherefore, if you add its Complement to two right Angles, *viz.* the Angle SEI to the Angle SEF, there will be given the Angle IEF. Therefore there are given the Angles of the Triangle IEF, together with the Side EF, and consequently there is given the Side IE. And by a like Argument there are given KE and LE. There are given therefore in Position the four Lines AI, BI, CK, DL, and consequently the Problem comes to this, that four Lines being given in Position, we may find a fifth, which shall be cut by these four in a given Ratio.

Having let fall to AI the Perpendiculars BM, CN, DO, by reason of the given Angle AIB there is given the Ratio of BM to MI. But BM to CN is in the given Ratio of BA and CA, and by reason of the given Angle CKN there is given the Ratio of CN to KN. Wherefore, there is also given the Ratio of BM to KN; and thence also the Ratio of BM to $MI - KN$, that is, to $MN + IK$. Take P to IK as is AB to BC, and since MA is to MN in the same Ratio, $P + MA$ will be to $IK + MN$ in the same Ratio, that is, in a given Ratio. Wherefore, there is given the Ratio of BM to $P + MA$. And by a like Argument, if Q be taken to IL in the Ratio of AB to BD, there will be given the Ratio of BM to $Q + MA$. And then the Ratio of BM to the Difference of $P + MA$ and $Q + MA$ will be given. But that Difference, *viz.* $P - Q$, or $Q - P$ is given, and then there will be given BM. But BM being given, there are alſo given $P + MA$ and MI, and thence, MA, ME, AE, and the Angle EAB.

Theſe being found, erect at A a Line perpendicular to the Plan of the Ecliptick, which ſhall be to the Line EA as the Tangent of the Comet's Latitude in the firſt Obſervation to Radius, and the End of that Perpendicular will be the Planet's Place in the firſt Obſervation. Whence the Diſtance of the Comet from the Earth is given in the Time of that Obſervation.

And after the ſame Manner, if from the Point B you erect a Perpendicular which ſhall be to the Line BF as the Tangent of the Comet's Latitude in the ſecond Obſervation to Radius, you'll have the Place of the Comet's Center in that ſecond Obſervation, and a Line drawn from the firſt Place to the ſecond, is that in which the Comet moves through the Heaven.

PROBLEM LIII.

If the given Angle CAD move about the angular Point A given in Poſition, and the given Angle CBD about the angular Point B given alſo in Poſition, on this Condition, that the Legs AD, BD, ſhall always cut one another in the right Line EF given likewiſe in Poſition; to find the Curve, which the Interſection C of the other Legs AC, BC, deſcribes. [Vide Figure 74.]

PRODUCE CA to d, ſo that Ad ſhall be $= AD$, and produce CB to δ, ſo that $B\delta$ ſhall be $=$ to BD. Make the Angle Ade equal to the Angle ADE, and the Angle $B\delta f$ equal to the Angle BDF, and produce AB on both Sides till it meet de and δf in e and f. Produce alſo ed to G, that dG ſhall be $= \delta f$, and from the Point C to the Line AB draw CH parallel to ed, and CK parallel to $f\delta$. And conceiving the Lines eG, $f\delta$ to remain immoveable

able while the Angles CAD, CBD, move by the aforesaid Law about the Poles A and B, Gd will always be equal to $f\delta$, and the Triangle CHK will be given in Specie. Make therefore $Ae=a$, $eG=b$, $Bf=c$, $AB=m$, $BK=x$, and $CK=y$. And BK will be $: CK :: Bf : f\delta$. Therefore $f\delta = \frac{cy}{x} = Gd$. Take this from Ge, and there will remain $ed = b - \frac{cy}{x}$. Since the Triangle CKH is given in Specie, make $CK : CH :: d : e$, and $CH : HK :: d : f$; and CH will $= \frac{ey}{d}$, and $HK = \frac{fy}{d}$. And consequently $AH = m - x - \frac{fy}{d}$. But $AH : HC :: Ae : ed$, that is, $m - x - \frac{f}{d}y : \frac{ey}{d} :: a : b - \frac{cy}{x}$. Therefore, by multiplying the Means and Extreams together, there will be made $mb - \frac{mcy}{x} - bx + cy - \frac{bf}{d}y + \frac{cfyy}{dx} = \frac{aey}{d}$. Multiply all the Terms by dx, and reduce them into Order, and there will come out $fcyy \begin{matrix} +dc \\ -ae \\ -fb \end{matrix} xy - dcmy - bdxx + bdmx = 0$. Where, since the unknown Quantities x and y ascend only to two Dimensions, it is evident, that the Curve Line that the Point C describes is a Conick Section. Make $\frac{ae + fb - dc}{c} = 2p$, and there will come out $yy = \frac{2pxy}{f} + \frac{dm}{f}y + \frac{bd}{fc}xx - \frac{bdm}{fc}x$. And the Square Root being extracted, $y = \frac{p}{f}x + \frac{dm}{2f} \pm$

$$\sqrt{\frac{pp}{ff}xx + \frac{bd}{fc}xx + \frac{pdm}{ff}x - \frac{bdm}{fc}x + \frac{ddmm}{4ff}}.$$

Whence we infer, that the Curve is an Hyperbola, if $\frac{bd}{fc}$ be Affirmative, or Negative and not greater than $\frac{pp}{ff}$; and

 a Pa-a Parabola, if $\frac{bd}{fc}$ be Negative and equal to $\frac{pp}{ff}$; an Ellipse or a Circle, if $\frac{bd}{fc}$ be both Negative and greater than $\frac{pp}{ff}$. Q.E.I.

PROBLEM LIV.

To describe a Parabola which shall pass through four Points given. [Vide Figure 75.]

LET those given Points be A, B, C, D. Join AB, and bisect it in E. And through E draw VE, a right Line, which conceive to be the Diameter of a Parabola, the Point V being its Vertex. Join AC, and draw DG parallel to AB, and meeting AC in G. Make $AB = a$, $AC = b$, $AG = c$, $GD = d$. Upon AC take AP of any Length, and from P draw PQ parallel to AB, and conceiving Q to be a Point of the Parabola; make $AP = x$, $PQ = y$. And take any Æquation expressive of a Parabola, which determines the Relation between AP and PQ. As that y is $= e + fx \pm \sqrt{gg + hx}$.

Now if AP or x be put $= 0$, the Point P falling upon A, PQ or y will be $= 0$, as also $= -AB$. And by writing in the assum'd Æquation 0 for x, you'll have $y = e \pm \sqrt{gg}$, that is, $= e \pm g$. The greater of which Values of y, $e + g$ is $= 0$, the lesser $e - g = -AB$, or to $-a$. Therefore $e = -g$, and $e - g$, that is, $-2g = -a$, or $g = \frac{1}{2}a$. And so in room of the assum'd Æquation you'll have this $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + hx}$.

Moreover, if AP or x be made $= AC$, so that the Point P falls upon C, you'll have again $PQ = 0$. For x therefore in the last Æquation write AC or b, and for y write 0; and you'll have $0 = -\frac{1}{2}a + fb + \sqrt{\frac{1}{4}aa + hb}$, or $\frac{1}{2}a - fb = \sqrt{\frac{1}{4}aa + hb}$; and the Parts being squar'd $-afb + ffbb = hb$, or $ffb - fa = h$. And so, in room of the assum'd Æquation, there will be had this, $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + ffbx - fax}$.

Moreover, if AP or x be made $= AG$ or c, PQ or y will be $= -GD$ or $-d$. Wherefore, for x and y in the last Æquation write c and $-d$, and you'll have $-d = -\frac{1}{2}a$

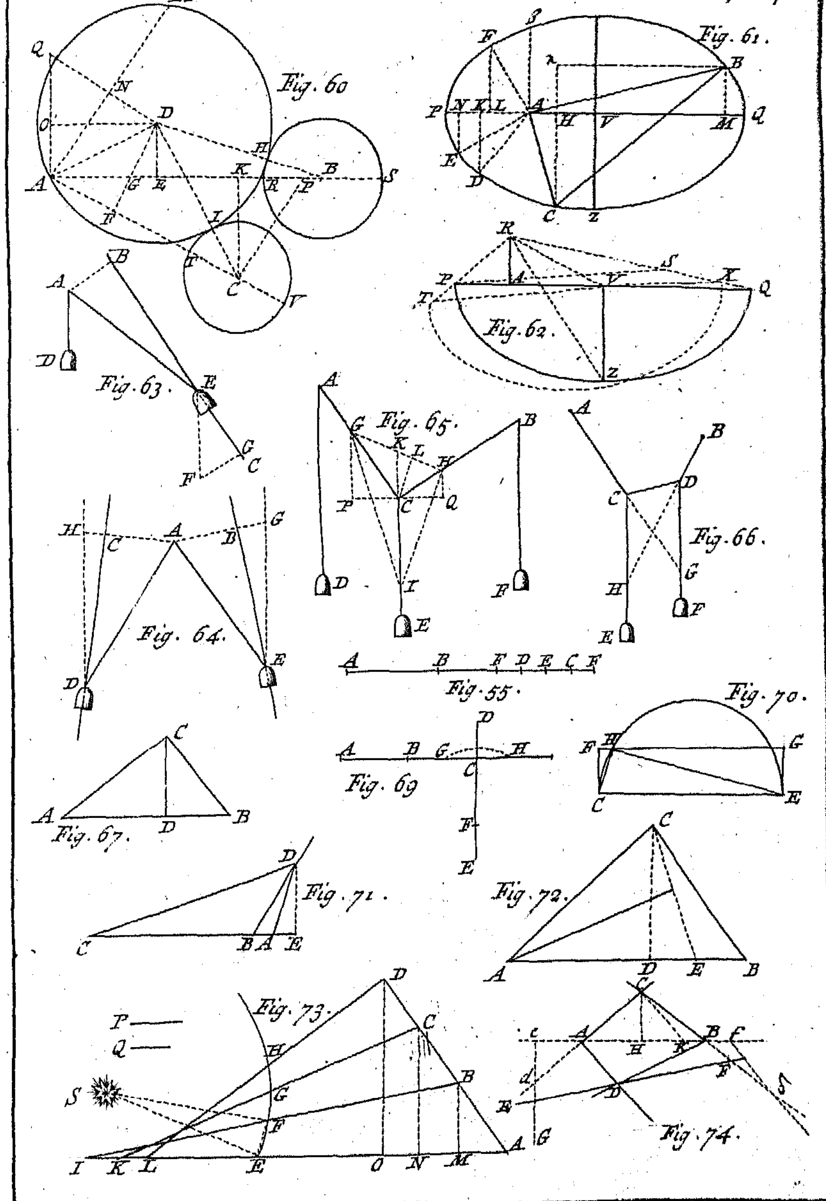
Fig. 60
Fig. 61.
Fig. 62.
Fig. 63.
Fig. 65.
Fig. 66.
Fig. 64.
Fig. 55.
Fig. 70.
Fig. 69
Fig. 67.
Fig. 71.
Fig. 72.
Fig. 73.
Fig. 74.

$\frac{1}{2}a + fc = \sqrt{\frac{1}{4}aa + ffbc - fac}$, or $\frac{1}{2}a - d - fc = \sqrt{\frac{1}{4}aa + ffbc - fac}$. And the Parts being squar'd, $-ad - fac + dd + 2dcf + ccff = ffbc - fac$. And the Æquation being order'd and reduc'd, $ff = \frac{2d}{b-c}f + \frac{dd-ad}{bc-cc}$. For $b-c$, that is, for GC write k, and that Æquation will become $ff = \frac{2d}{k}f + \frac{dd-ad}{kc}$. And the Root being extracted, $f = \frac{d}{k} \pm \sqrt{\frac{ddc + ddk - adk}{kkc}}$. But f being found, the Parabolick Æquation, *viz.* $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + ffbx - fax}$ will be fully determin'd; by whose Construction therefore the Parabola will also be determin'd. The Construction is thus: Draw CH parallel to BD meeting DG in H. Between DG and DH take a mean Proportional DK, and draw EI parallel to CK, bisecting AB in E, and meeting DG in I. Then produce IE to V, so that EV shall be to $EI :: EBq : DIq - EBq$, and V will be the Vertex, VE the Diameter, and $\frac{BEq}{VE}$ the *Latus Rectum* of the Parabola sought.

PROBLEM LV.

To describe a Conick Section through five Points given. [Vide Figure 76.]

LET those Points be A, B, C, D, E. Join AC, BE, cutting one another in H. Draw DI parallel to BE, and meeting AC in I. As also EK parallel to AC, and meeting DI produc'd in K. Produce ID to F, and EK to G; so that AHC shall be : $BHE :: AIC : FID :: EKG : FKD$, and the Points F and G will be in a Conick Section, as is known.

But you ought to observe this, if the Point H falls between all the Points A, C, and B, E, or without them all, the Point I must either fall between all the Points A, C, and F, D, or without them all; and the Point K between all the Points D, F, and E, G, or without them all. But if the Point H falls between the two Points A, C, and without the other two B, E, or between those two BE, and without

out the other two AC, the Point I ought to fall between two of the Points A, C and F, D, and without the other two of them; and in like Manner, the Point K ought to fall between two of the Points D, F, and E, G, and without Side of the two other of them; which will be done by taking IF, KG, on this or that Side of the Points I, K, according to the Exigency of the Problem. Having found the Points F and G, bisect AC and EG in N and O; also BE, FD in L and M. Join NO, LM, cutting one another in R; and LM and NO will be the Diameters of the Conick Section, R its Center, and BL, FM, Ordinates to the Diameter LM. Produce LM on both Sides, if there be Occasion, to P and Q, so that BLq shall be to $FMq :: PLQ : PMQ$, and P and Q will be the Vertex's of the Conick Section, and PQ the *Latus Transversum*. Make $PLQ : LBq :: PQ : T$, and T will be the *Latus Rectum*. Which being known, the Figure is known.

It remains only that we may shew how LM is to be produc'd each Way to P and Q, so that BLq may be : $FMq :: PLQ : PMQ$, *viz.* PLQ, or $PL \times LQ$, is $\overline{PR-LR} \times \overline{PR+LR}$; for PL is $PR-LR$, and LQ is $RQ+LR$, or $PR+LR$. Moreover, $\overline{PR-LR} \times \overline{PR+LR}$, by multiplying, becomes $PRq-LRq$. And after the same Manner, PMq is $\overline{PR+RM} \times \overline{PR-RM}$, or $PRq-RMq$. Therefore $BLq : FMq :: PRq - LRq : PRq-RMq$; and by dividing, $BLq-FMq : FMq :: RMq-LRq : PRq-RMq$. Wherefore since there are given $BLq-FMq$, FMq and $RMq-LRq$, there will be given $PRq-RMq$. Add the given Quantity RMq, and there will be given the Sum PRq, and consequently its Root PR, to which QR is equal.

PRO-

PROBLEM LVI.

To describe a Conick Section which shall pass through four given Points, and in one of those Points shall touch a right Line given in Position. [Vide Figure 77.]

LET the four given Points be A, B, C, D, and the right Line given in Position be AE, which let the Conick Section touch in the Point A. Join any two Points D, C, and let DC produc'd, if there be Occasion for it, meet the Tangent in E. Through the fourth Point B draw BF parallel to DC, which shall meet the same Tangent in F. Also draw DI parallel to the Tangent, and which may meet BF in I. Upon FB, DI, produc'd, if there be Occasion, take FG, HI, of such Length as $AEq : CED :: AFq : BFG :: DIH : BIG$. And the Points G and H will be in a Conick Section as is known; if you only take FG, IH, on the right Sides of the Points F and I, according to the Rule deliver'd in the former Problem. Bisect BG, DC, DH, in K, L, and M. Join KL, AM, cutting one another in O, and O will be the Center, A the Vertex, and HM an Ordinate to the Semi-Diameter AO; which being known, the Figure is known.

PROBLEM LVII.

To describe a Conick Section which shall pass through three given Points, and touch right Lines given in Position in two of those Points. [Vide Figure 78.]

LET those given Points be A, B, C, touching AD, BD, in the Points A and B, and let D be the common Intersection of those Tangents. Bisect AB in E. Draw DE, and produce it till in F it meets CF drawn parallel to AB; and DF will be the Diameter, and AE and CF the Ordinates to [that] Diameter. Produce DF to O, and on DO take OV a mean Proportional between DO and EO, on this Condition, that also $AEq : CFq :: VE \times \overline{VO + OE} : VF$

$: VF \times \overline{VO + OF}$; and V will be the Vertex, and O the Center of the Figure. Which being known, the Figure will alſo be known. But VE is $= VO - OE$, and conſequently $VE \times \overline{VO + OE} = \overline{VO - OE} \times \overline{VO + OE} = VOq - OEq$. Beſides, becauſe VO is a mean Proportional between DO and EO, VOq will be $= DOE$, and conſequently $VOq - OEq = DOE - OEq = DEO$. And by a like Argument you'll have $VF \times \overline{VO + OF} = VOq - OFq = DOE - OFq$. Therefore $AEq : CFq :: DEO : DOE - OFq$. OFq is $= EOq - 2FEO + FEq$. And conſequently $DOE - OFq = DOE - OEq + 2FEO - FEq = DEO + 2FEO - FEq$. And $AEq : CFq :: DEO : DEO + 2FEO - FEq :: DE : DE + 2FE - \frac{FEq}{EO}$. Therefore there is given $DE + 2FE - \frac{FEq}{EO}$. Take away from this given Quantity $DE + 2FE$, and there will remain $\frac{FEq}{EO}$ given. Call that N; and $\frac{FEq}{N}$ will be $= EO$, and conſequently EO will be given. But EO being given, there is alſo given VO, the mean Proportional between DO and EO.

After this Way, by ſome of *Apollonius*'s Theorems, theſe Problems are expeditiouſly enough ſolv'd; which yet may be ſolv'd by Algebra without thoſe Theorems. As if the firſt of the three laſt Problems be propos'd: [*Vide Figure* 78.] Let the five given Points be A, B, C, D, E, through which the Conick Section is to paſs. Join any two of them, A, C, and any other two, B, E, by Lines cutting (or interſecting) one another in H. Draw DI parallel to BE meeting AC in I; as alſo any other right Line KL meeting AC in K, and the Conick Section in L. And imagine the Conick Section to be given, ſo that the Point K being known, there will at the ſame Time be known the Point L; and making $AK = x$, and $KL = y$, to expreſs the Relation between x and y, aſſume any Æquation which generally expreſſes the Conick Sections; ſuppoſe this, $a + bx + cxx + dy + exy + yy = 0$. Wherein a, b, c, d, e, denote determinate Quantities with their Signs, but x and y indeterminate Quantities. Now if we can find the determinate Quantities a, b, c, d, e, the Conick Section will be known. If therefore the Point L falls upon the Point A, in that Caſe AK and KL, that is, x and y, will be 0. Then all the Terms of the

Æquation

Æquation besides a will vanish, and there will remain $a=0$. Wherefore a is to be blotted out in that Æquation, and the other Terms $bx+cxx+dy+exy+yy$ will be $=0$. But if L falls upon C, AK, or x, will be $=AC$, and LK or y, $=0$. Put therefore $AC=f$, and by substituting f for x and 0 for y, the Æquation for the Curve $bx+cxx+dy+exy+yy=0$, will become $bf+cff=0$, or $b=-cf$. And having writ in that Æquation $-cf$ for b, it will become $-cfx+cxx+dy+exy+yy=0$. Moreover, if the Point L falls upon the Point B, AK or x will be $=AH$, and KL or $y=BH$. Put therefore $AH=g$, and $BH=h$, and then write g for x and h for y, and the Æquation $-cfx+cxx$, &c. will become $-cfg+cgg+dh+egh+hh=0$. Now if the Point L falls upon E, AK will be $=AH$, or $x=g$, and KL or $y=HE$. For HE therefore write $-k$, with a Negative Sign, because HE lies on the contrary Side of the Line AC, and by substituting g for x and $-k$ for y, the Æquation $-cfx+cxx$, &c. will become $-cfx+cgg-dk-egk+kk=0$. Take away this from the former Æquation $-cfg+cgg+dh+egh+hh$, and there will remain $dh+egh+hh+dk+egk-kk=0$. Divide this by $h+k$, and there will come out $d+eg+h-k=0$. Take away this multiply'd by h from $-cfg+cgg+dh+egh+hh=0$, and there will remain $-cfg+cgg+hk=0$, or $\frac{hk}{-gg+fg}=c$. Lastly, if the Point L falls upon the Point D, AK or x will be $=AI$, and KL or y will be $=ID$. Wherefore, for AI write m, and for ID, n, and likewise for x and y substitute m and n, and the Æquation $-cfx+cxx$, &c. will become $-cfm+cmm+dn+emn+nn=0$. Divide this by n, and there will come out $\frac{-cfm+cmm}{n}+d+em+n=0$. Take away $d+eg+h-k=0$, and there will remain $\frac{-cfm+cmm}{n}+em-eg+n-h+k=0$, or $\frac{cmm-cfm}{n}+n-h+k=eg-em$. But now by reason of the given Points A, B, C, D, E, there are given AC, AH, AI, BH, EH, DI, that is, f, g, m, h, k, n. And consequently by the Æ-

 quation

Æquation $\frac{hk}{fg-gg}=c$ there is given C. But c being given by the Æquation $\frac{cmm-cfm}{n}+n-h+k=eg-em$ there is given $eg-em$. Divide this given Quantity by the given one $g-m$, and there will come out the given e. Which being found, the Æquation $d+eg+h-k=0$, or $d=k-h-eg$, will give d. And these being known, there will at the same Time be determin'd the Æquation expressive of the Conick Section sought, *viz.* $cfx=cxx+dy+exy+yy$. And from that Æquation, by the Method of *Des Cartes*, the Conick Section will be determin'd.

Now if the four Points A, B, C, E, and the Position of the right Line AF, which touches the Conick Section in one of those Points, A were given, the Conick Section may be thus more easily determin'd. Having found, as above, the Æquations $cfx=cxx+dy+exy+yy$, $d=k-h-eg$, and $c=\frac{hk}{fg-gg}$, conceive the Tangent AF to meet the right Line EH in F, and then the Point L to be moved along the Perimeter of the Figure CDE till it fall upon the Point A; and the ultimate Ratio of LK to AK will be the Ratio of FH to AH, as will be evident to any one that contemplates the Figure. Make $FH=p$, and in this Case where LK, AK, are in a vanishing State, you'll have $p:g::y:x$, or $\frac{gy}{p}=x$. Wherefore for x, in the Æquation $cfx=cxx+dy+exy+yy$, write $\frac{gy}{p}$, and there will arise $\frac{cfgy}{p}=\frac{cggyy}{pp}+dy+\frac{egyy}{p}+yy$. Divide all by y, and there will come out $\frac{cfg}{p}=\frac{cggy}{pp}+d+\frac{egy}{p}+y$. Now because the Point L is suppos'd to fall upon the Point A, and consequently KL, or y, to be infinitely small or nothing, blot out the Terms which are multiply'd by y, and there will remain $\frac{cfg}{p}=d$. Wherefore make $\frac{hk}{fg-gg}=c$, then $\frac{cfg}{p}=d$. Lastly, $\frac{k-h-d}{g}=e$, and

having

having found c, d, and e, the Æquation $cfx = cxx + dy + exy + yy$ will determine the Conick Section.

If, lastly, there are only given the three Points A, B, C, together with the Position of the two right Lines AT, CT, which touch the Conick Section in two of those Points, A and C, there will be obtain'd, as above, this Æquation expressive of a Conick Section, $cfx = cxx + dy + exy + yy$. [*Vide Figure* 80.] Then if you suppose the Ordinate KL to be parallel to the Tangent AT, and it be conceiv'd to be produc'd, till it again meets the Conick Section in M, and that Line LM to approach to the Tangent AT till it coincides with it at A, the ultimate [or evanescent] Ratio of the Lines KL and KM to one another, will be a Ratio of Æquality, as will appear to any one that contemplates the Figure. Wherefore in that Case KL and KM being equal to each other, that is, the two Values of y, (*viz.* the Affirmative one KL, and the Negative one KM) being equal, those Terms of the Æquation ($cfx = cxx + dy + exy + yy$) in which y is of an odd Dimension, that is, the Terms $dy + exy$ in respect of the Term yy, wherein y is of an even Dimension, will vanish. For otherwise the two Values of y, *viz.* the Affirmative and the Negative, cannot be equal; and in that Case AK is infinitely less than LK, that is x than y, and consequently the Term exy than the Term yy. And consequently being infinitely less, may be reckon'd for nothing. But the Term dy, in respect of the Term yy, will not vanish as it ought to do, but will grow so much the greater, unless d be suppos'd to be nothing. Therefore the Term dy is to be blotted out, and so there will remain $cfx = cxx + exy + yy$, an Æquation expressive of a Conick Section. Conceive now the Tangents AT, CT, to meet one another in T, and the Point L to come to approach to the Point C, till it coincides with it. And the ultimate Ratio of KL to KC will be that of AT to AC. KL was y; AK, x; and AC, f; and consequently KC, $f - x$; make $AT = g$, and the ultimate Ratio of y to $f - x$, will be the same as of g to f. The Æquation $cfx = cxx + exy + yy$, subtracting on both Sides cxx, becomes $cfx - cxx = exy + yy$, that is, $\overline{f - x}$ into $cx = y$ into $\overline{ex + y}$. Therefore $y : f - x :: cx : ex + y$, and consequently $g : f :: cx : ex + y$. But the Point L falling upon C, y becomes nothing. Therefore $g : f :: cx : ex$. Divide the latter Ratio by x, and it will become $g : f :: c : e$, and

and $\frac{cf}{g} = e$. Wherefore, if in the Æquation $cfx = cxx + exy + yy$, you write $\frac{cf}{g}$ for e, it will become $cfx = cxx + \frac{cf}{g}xy + yy$, an Æquation expreſſive of a Conick Section. Laſtly, draw BH parallel to KL, or AT, from the given Point B, through which the Conick Section ought to paſs, and which ſhall meet AC in H, and conceiving KL to come towards BH, till it coincides with it, in that Caſe AH will be $= x$, and $BH = y$. Call therefore the given $AH = m$, and the given $BH = n$, and then for x and y, in the Æquation $cfx = cxx + \frac{cf}{g}xy + yy$, write m and n, and there will ariſe $cfm = cmm + \frac{cf}{g}mn + nn$. Take away on both Sides $cmm + \frac{cf}{g}mn$, and there will come out $cfm - cmm - \frac{cf}{g}mn = nn$. Put $f - m - \frac{fn}{g} = s$, and csm will be $= nn$. Divide each Part of the Æquation by sm, and there will ariſe $c = \frac{nn}{sm}$. But having found c, the Æquation for the Conick Section is determin'd ($cfx = cxx + \frac{cf}{g}xy + yy$). And then, by the Method of *Des Cartes*, the Conick Section is given, and may be deſcrib'd.

PRO-

Problem LVIII.

Having given the Globe A, and the Position of the Wall DE, and BD the Distance of the Center of the Globe B from the Wall; to find the Bulk of the Globe B, on this Condition, that if the Globe A, (whose Center is in the Line BD, which is perpendicular to the Wall, and produc'd out beyond B) be moved in free absolute Space, and where Gravity can't act, with an uniform Motion towards D, till it falls upon [or strikes against] the other quiescent Globe B; and that Globe B, after it is reflected from the Wall, shall meet the Globe A in the given Point C. [Vide Figure 81.]

LET the Velocity of the Globe A before Reflection be a, and by *Problem* 12. the Velocity of the Globe A will be after Reflection $=\frac{aA-aB}{A+B}$, and the Velocity of the Globe B after Reflection will be $=\frac{2aA}{A+B}$. Therefore the Velocity of the Globe A to the Velocity of the Globe B is as $A-B$ to $2A$. On GD take $gD=GH$, *viz.* to the Diameter of the Globe B, and those Velocities will be as GC to $Gg+gC$. For when the Globe A struck upon the Globe B, the Point G, which being on the Surface of the Globe B is moved in the Line AD, will go through the Space Gg before that Globe B shall strike against the Wall, and through the Space gC after it is reflected from the Wall; that is, through the whole Space $Gg+gC$, in the same Time wherein the Point F of the Globe A shall pass through the Space GC, so that both Globes may again meet and strike one another in the given Point C. Wherefore, since the Intervals BC and CD are given, make $BC=m$, $BD+CD=n$, and $BG=x$, and GC will be $=m+x$, and $Gg+gC=GD+DC-2gD=GB+BD+DC-2GH=x+n-4x$, or $=n-3x$. Above you had $A-B$ to $2A$, as the Velocity of the Globe A to the Velocity of the Globe

Globe B, and the Velocity of the Globe A to the Velocity of the Globe B, as GC to $Gg+gC$, and consequently $A-B$ to $2A$, as GC to $Gg+gC$; therefore since GC is $=m+x$, and $Gg+gC=n-3x$, $A-B$ will be to $2A$ as $m+x$ to $n-3x$. Moreover, the Globe A is to the Globe B as the Cube of its Radius AF to the Cube of the others Radius GB; that is, if you make the Radius AF to be s, as s^3 to x^3; therefore $s^3-x^3 : 2s^3$ ($:: A-B : 2A$) $:: m+x : n-3x$. And multiplying the Means and Extreams by one another, you'll have this Æquation, $s^3n-3s^3x-nx^3+3x^4=2ms^3+2sx^3$. And by Reduction $3x^4-nx^3-5s^3x\begin{array}{l}+s^3n\\-2s^3m\end{array}=0$. From the Construction of which Æquation there will be given x, the Semi-Diameter of the Globe B; which being given, that Globe is also given. Q. E. F.

But note, when the Point C lies on contrary Sides of the Globe B, the Sign of the Quantity $2m$ must be chang'd, and written $3x^4-nx^3-5s^3x\begin{array}{l}+s^3n\\+2s^3m\end{array}=0$.

If the Globe B were given, and the Globe A sought on this Condition, that the two Globes, after Reflection, should meet in C, the Question would be easier; *viz.* in the last Æquation found, x would be suppos'd to be given, and s to be sought. Whereby, by a due Reduction of that Æquation, the Terms $-5s^3x+s^3n-2s^3m$ being translated to the contrary Side of the Æquation, and each Part divided by $5x-n+2m$, there would come out $\frac{3x^4-nx^3}{5x-n+2m}=s^3$. Where s will be obtain'd by the bare Extraction of the Cube Root.

Now if both Globes being given, you were to find the Point C, in which both would fall upon one another after Reflection, the same Æquation by due Reduction would give $m=\frac{1}{2}n-\frac{5}{2}x+\frac{3x^4-x^3n}{2s^3}$; that is, $BC=\frac{1}{2}Hg+\frac{1}{2}gC-\frac{B}{2A}\times\overline{HD+DC}$. For above, $n-3x$ was $=Gg+gC$. Whence, if you take away $2x$, or GH, there will remain $n-5x=Hg+gC$. The Half whereof is $\frac{1}{2}n-\frac{5}{2}x=\frac{1}{2}Hg+\frac{1}{2}gC$. Moreover, from n, or $BD+CD$, take away x, or BH, and there will remain $n-x$, or

$\overline{HD+CD}$. Whence, since $\frac{x^3}{2s^3}=\frac{B}{2A}$, you'll have $\frac{x^3}{2s^3}\times\overline{n-x}$, or $\frac{nx^3-x^4}{2s^3}=\frac{B}{2A}\times\overline{HD+CD}$. And the Signs being chang'd, $\frac{x^4-nx^3}{2s^3}=-\frac{B}{2A}\times\overline{HD+CD}$.

PROBLEM LIX.

If two Globes, A and B, are join'd together by a small Thread PQ, and the Globe B hanging on the Globe A; if you let fall the Globe A, so that both Globes may begin to fall together by the sole Force of Gravity in the same perpendicular Line PQ; and then the lower Globe B, after it is reflected upwards from the Bottom or Horizontal Plane FG, it shall meet the upper Globe A, as falling, in a certain Point D; from the given Length of the Thread PQ, and the Distance DF of that Point D from the Bottom, to find the Height PF, from which the upper Globe A ought to be let fall to [cause] this Effect. [Vide Figure 83.]

LET a be the Length of the Thread PQ. In the Perpendicular $PQRF$, from F upwards take FE equal to QR the Diameter of the lower Globe, so that when the lowest Point R of that Globe falls upon the Bottom in F, its upper Point Q shall possess the Place E; and let ED be the Distance through which that Globe, after it is reflected from the Bottom, shall, by ascending, pass, before it meets the upper falling Globe in the Point D. Therefore, by reason of the given Distance DF of the Point D from the Bottom, and the Diameter EF of the inferiour Globe, there will be given their Difference DE. Let that $=b$, and let the Depth RF, or QE, through which that lower Globe by falling before it touches the Bottom be $=x$, if it be unknown. And having found x, if to it you add EF and PQ,

PQ; there will be had the Height PF, from which the upper Globe ought to fall to have the desir'd Effect.

Since therefore PQ is $=a$, and $QE=x$, PE will be $=a+x$. Take away DE or b, and there will remain $PD=a+x-b$. But the Time of the Descent of the Globe A is as the Root of the Space describ'd in falling, or $\sqrt{a+x-b}$, and the Time of the Descent of the other Globe B as the Root of the Space describ'd by [its] falling, or $\sqrt{x}$, and the Time of its Ascent as the Difference of that Root, and of the Root of the Space which it would describe by falling only from Q to D. For this Difference is as the Time of Descent from D to E, which is equal to the Time of Ascent from E to D. But that Difference is $\sqrt{x}-\sqrt{x-b}$. Whence the Time of Descent and Ascent together will be as $2\sqrt{x}-\sqrt{x-b}$. Wherefore, since this Time is equal to the Time of Descent of the upper Globe, the $\sqrt{a+x-b}$ will be $=2\sqrt{x}-\sqrt{x-b}$. The Parts of which Æquation being squar'd, you'll have $a+x-b=5x-b-4\sqrt{xx-bx}$, or $a=4x-4\sqrt{xx-bx}$; and the Æquation being order'd, $4x-a=4\sqrt{xx-bx}$; and squaring the Parts of that Æquation again, there arises $16xx-8ax+aa=16xx-16bx$, or $aa=8ax-16bx$; and dividing all by $8a-16b$, you'll have $\frac{aa}{8a-16b}=x$. Make therefore as $8a-16b$ to a, so a to x, and you'll have x or QE. Q. E. I.

Now if from the given QE you are to find the Length of the Thread PQ or a; the same Æquation $aa=8ax-16bx$, by extracting the affected Quadratick Root, would give $a=4x-\sqrt{16xx-16bx}$; that is, if you take QY a mean Proportional between QD and QE, PQ will be $=4EY$. For that mean Proportional will be $\sqrt{x\times\overline{x-b}}$, or $\sqrt{xx-bx}$; which subtracted from x, or QE, leaves EY, the Quadruple whereof is $4x-4\sqrt{xx-bx}$.

But if from the given Quantities QE, or x, as also the Length of the Thread PQ, or a, there were sought the Point D on which the upper Globe falls upon the under one; the Distance DE, or b, of that Point from the given Point E, will be had from the precedent Æquation $aa=8ax-16bx$ by transferring aa and $16bx$ to the contrary Sides of the Æquation

Æquation with the Signs chang'd, and by dividing the whole by $16x$. There will arise $\frac{8ax-aa}{16x}=b$. Make therefore as $16x$ to $8x-a$, so a to b, and you'll have b or DE.

Hitherto I have suppos'd the Globes ty'd together by a small Thread to be let fall together. Which, if they are let fall at different Times connected by no Thread, so that the upper Globe A, for Example, being let fall first, shall descend through the Space PT before the other Globe begins to fall, and from the given Distances PT, PQ, and DE, you are to find the Height PF, from which the upper Globe ought to be let fall, so that it shall fall upon the inferior or lower one at the Point D. Make $PQ=a$, $DE=b$, $PT=c$, and $QE=x$, and PD will be $=a+x-b$, as above. And the Time wherein the upper Globe, by falling, will describe the Spaces PT and TD, and the lower Globe by falling before, and then by re-ascending, will describe the Sum of the Spaces $QE+ED$ will be as $\sqrt{PT}$, $\sqrt{PD}-\sqrt{PT}$, and $2\sqrt{QE}-\sqrt{QD}$; that is, as the $\sqrt{c}$, $\sqrt{a+x-b}-\sqrt{c}$, and $2\sqrt{x}-\sqrt{x-b}$, but the two last Times, because the Spaces TD and $QE+ED$ are describ'd together, are equal. Therefore $\sqrt{a+x-b}-\sqrt{c}=2\sqrt{x}-\sqrt{x-b}$. And the Parts being squar'd $a+c-2\sqrt{ca+cx-cb}=4x-4\sqrt{xx-bx}$. Make $a+c=e$, and $a-b=f$, and by a due Reduction $4x-e+2\sqrt{cf+cx}=4\sqrt{xx-bx}$, and the Parts being squar'd $ee-8ex+16xx+4cf+4cx+\overline{16x-4e}\times\sqrt{cf+cx}=16xx-16bx$. And blotting out on both Sides $16xx$, and writing m for $ee+4ef$, and also writing n for $8e-16b-4c$, you'll have by due Reduction $\overline{16x-4e}\times\sqrt{cf+cx}=nx-m$. And the Parts being squar'd [you'll have] $256cfxx+256cx^3-128cefx-128cexx+16ceef+16ceex=nnxx-2mnx+mm$. And having order'd the Æquation

$$256cx^3 \begin{array}{l} +256cf \\ -128ce \\ -nn \end{array} xx \begin{array}{l} -128cef \\ +16cee \\ +2mn \end{array} x \begin{array}{l} +6ceef \\ -mm \end{array} =0.$$

By the Construction of which Æquation x or QE will be given, to which if you add the given Distances PQ and EF, you'll have the Height PF, which was to be found.

Problem LX.

If two quiescent Globes, the upper one A and the under one B, are let fall at different Times; and the lower Globe begins to fall in the same Moment that the upper one, by falling, has describ'd the Space PT; to find the Places α, β, which those falling Globes shall occupy when their Interval or Distance $\pi\chi$ is given. [Vide Figure 84.]

SINCE the Distances PT, PQ, and $\pi\chi$ are given, call the first a, the second b, the third c, and for $P\pi$, or the Space that the superior Globe describes by falling before it comes to the Place sought α, put x. Now the Times wherein the upper Globe describes the Spaces PT, $P\pi$, $T\pi$, and the lower one the Space $Q\chi$, are as $\sqrt{PT}$, $\sqrt{P\pi}$, $\sqrt{P\pi} - \sqrt{PT}$, and $\sqrt{Q\chi}$. The latter two of which Times, because the Globes by falling together describe the Spaces $T\pi$ and $Q\chi$, are equal. Whence also the $\sqrt{P\pi} - \sqrt{PT}$ will be equal to the $\sqrt{Q\chi}$. $P\pi$ was $= x$, and $PT = a$, and by adding $\pi\chi$, or c, to $P\pi$, and subtracting PQ, or b, from the Sum you'll have $Q\chi = x + c - b$. Wherefore substituting these, you'll have $\sqrt{x} - \sqrt{a} = \sqrt{x + c - b}$. And squaring both Sides of the Æquation, there will arise $x + a - 2\sqrt{ax} = x + c - b$. And blotting out on both Sides x, and ordering the Æquation, you'll have $a + b - c = 2\sqrt{ax}$. And having squar'd the Parts, the Square of $a + b - c$ will be $= 4ax$, and that Square divided by $4a$ will be $= x$, or $4a$ will be to $a + b - c$ as $a + b - c$ is to x. But from x found, or $P\pi$, there is given the Place sought, *viz.* α of the superior Globe sought. And by the Distance of the Places, there is also given the Place of the lower one β.

And hence, if you were to find the Point where the upper Globe, by falling, will at length fall upon the lower one; by putting the Distance $\pi\chi = 0$, or by extirpating c, say, $4a$ is to $a + b$ as $a + b$ is to x, or $P\pi$, and the Point π will be that sought.

And

And reciprocally, if that Point π, or x, in which the upper Globe falls upon the under one, be given, and you are to find the Place T which the lower Point P of the upper falling Globe possess'd, or was then in, when the lower Globe began to fall; because $4a$ is to $a+b$ as $a+b$ is to x; or multiplying the Means and Extreams together, $4ax = aa + 2ab + bb$, and by due ordering of the Æquation $aa = 4ax - 2ab - bb$; extract the Square Root, and you'll have $a = 2x - b - 2\sqrt{xx - bx}$. Take therefore $V\pi$, a mean Proportional between $P\pi$ and $Q\pi$, and towards V take $VT = VQ$, and T will be the Point you seek. For $V\pi$ will be $= \sqrt{P\pi \times Q\pi}$, that is, $= \sqrt{x \times \overline{x - b}}$, or $= \sqrt{xx - bx}$; the double whereof subtracted from $2x - b$, or from $2P\pi - PQ$, that is, from $PQ + 2Q\pi$, leaves $PQ - 2VQ$, or $PV - VQ$ that is, PT.

If, lastly, the lower of the Globes, after the upper has fallen upon the lower, and the lower, by their Shock upon one another, is accelerated, and the superior one retarded, the Places are requir'd where, in falling, they shall acquire a Distance equal to a given right Line. In the first Place, you must seek the Place where the upper one falls upon the lower one; then from the known Magnitudes of the Globes, as also from their Celerities where they fall on each other, you must find the Celerities they shall have immediately after Reflection, after the same Way as in *Probl.* 12. Afterwards you must find the highest Places to which these Globes, if they were carry'd upwards, would ascend, and thence the Spaces will be known, which the Globes will describe by falling in [any] given Times after Reflection, as also the Difference of the Spaces; and reciprocally from that Difference assum'd, you may go back Analytically to the Spaces describ'd in falling.

As if the upper Globe falls upon the lower one at the Point π, [*Vide Figure* 85] and after Reflection, the Celerity of the upper one downwards be so great, as if it were upwards, it would cause that Globe to ascend through the Space πN; and the Celerity of the lower one downwards was so great, as that, if it were upwards, it would cause the lower one to ascend through the Space πM; then the Times wherein the upper Globe would reciprocally descend through the Spaces $N\pi$, NC, and the inferior one through the Spaces $M\pi$, MH, would be as $\sqrt{N\pi}$, $\sqrt{NC}$, $\sqrt{M\pi}$, $\sqrt{MH}$; and consequently the Times wherein the upper Globe would

 run

run the Space πG, and the lower one πH, would be as $\sqrt{NG} - \sqrt{N\pi}$, to $\sqrt{MH} - \sqrt{M\pi}$. Make those Times equal, and the $\sqrt{NG} - \sqrt{N\pi}$ will be $= \sqrt{MH} - \sqrt{M\pi}$. And, moreover, since there is given the Distance GH, put $\pi G + GH = \pi H$. And by the Reduction of these two Æquations, the Problem will be solv'd. As if $M\pi$ is $= a$, $N\pi = b$, $GH = c$, $\pi G = x$, you'll have, according to the latter Æquation, $x + c = \pi H$. Add $M\pi$, you'll have $MH = a + c + x$. To πG add $N\pi$, and you'll have $NG = b + x$. Which being found, according to the former Æquation, $\sqrt{b+x} - \sqrt{b}$ will be $= \sqrt{a+c+x} - \sqrt{a}$. Write e for $a + c$, and $\sqrt{f}$ for $\sqrt{a} + \sqrt{b}$, and the Æquation will be $\sqrt{b+x} = \sqrt{e+x} + \sqrt{f}$. And the Parts being squar'd $b + x = e + x + f + 2\sqrt{ef+fx}$, or $b - e - f = 2\sqrt{ef+fx}$. For $b - e - f$ write g, and you'll have $g = 2\sqrt{ef+fx}$, and the Parts being squar'd, $gg = 4ef + 4fx$, and by Reduction $\frac{gg}{4f} - e = x$.

PRO-

Problem LXI.

If there are two Globes, A, B, whereof the upper one A falling from the Height G, strikes upon another lower one B rebounding from the Ground H upwards; and these Globes so part from one another by Reflection, that the Globe A returns by Force of that Reflection to its former Altitude G, and that in the same Time that the lower Globe B returns to the Ground H; then the Globe A falls again, and strikes again upon the Globe B, rebounding again back from the Ground; and after this rate the Globes always rebound from one another and return to the same Place: From the given Magnitude of the Globes, the Position of the Ground, and the Place G from whence the upper Globe falls, to find the Place where the Globes shall strike upon each other. [Vide Figure 86.]

LET e be the Center of the Globe A, and f the Center of the Globe B, d the Center of the Place G wherein the upper Globe is in its greatest Height, g the Center of the Place of the lower Globe where it falls on the Ground, a the Semi-Diameter of the Globe A, b the Semi-Diameter of the Globe B, c the Point of Contact of the Globes falling upon one another, and H the Point of Contact of the lower Globe and the Ground. And the Swiftness of the Globe A, where it falls on the Globe B, will be the same which is generated by the Fall of the Globe from the Height de, and consequently is as $\sqrt{de}$. With this same Celerity the Globe A ought to be reflected upwards, that it may return to its former Place G. And the Globe B ought to be reflected with the same Celerity downwards wherewith it ascended, that it may return in the same Time to the Ground it had mounted up from. And that both these may come to pass, the Motion of the Globes in reflecting ought to be equal.

equal. But the Motions are compounded of the Celerities and Magnitudes together, and consequently the Product of the Bulk and Celerity of one Globe will be equal to the Product of the Bulk and Celerity of the other. Whence, if the Product of the Bulk and Celerity of one Globe be divided by the Bulk of the other Globe, you'll have the Celerity of the other before and after Reflection, or at the End of the Ascent, and at the Beginning of the Descent. Therefore this Celerity will be as $\frac{A\sqrt{de}}{B}$, or since the Globes are as the Cubes of the Radii as $\frac{a^3\sqrt{de}}{b^3}$. But as the Square of this Celerity to the Square of the Celerity of the Globe A just before Reflection, so would be the Height to which the Globe B would ascend with this Celerity, if it was not hinder'd by meeting the Globe A falling upon it, to the Height ed from which the Globe B descends. That is, as $\frac{Aq}{Bq}de$ to de, or as Aq to Bq, or a^6 to b^6, so that first Height to x, if you put x for the latter Height ed. Therefore this Height, *viz.* to which B would ascend, if it was not hinder'd, is $\frac{a^6}{b^6}x$. Let that be fK. To fK add fg, or $dH - ef - gH$; that is, $p - x$, if for the given $dH - ef - gH$ you write p, and x for the unknown de; and you'll have $Kg = \frac{a^6}{b^6}x + p - x$. Whence the Celerity of the Globe B, when it falls from K to the Ground, that is, when it falls through the Space Kg, which its Centre would describe in falling, will be as $\sqrt{\frac{a^6}{b^6}x + p - x}$. But that Globe falls from the Place Bcf to the Ground in the same Time that the upper Globe A ascends from the Place Ace to its greatest Height d, or on the other Hand falls from d to the Place Ace; and then since the Celerities of falling Bodies are equally augmented in equal Times, the Celerity of the Globe B, by falling to the Ground, will be augmented as much as is the whole Celerity which the Globe A acquires by falling in the same Time from d to e, or loses by ascending from e to d. Therefore, to the Celerity which the Globe B has in the Place Bcf, add the Celerity which the Globe

A

A has in the Place Acc, and the Sum, which is as $\sqrt{de} + \frac{a^3\sqrt{de}}{b^3}$, or $\sqrt{x} + \frac{a^3}{b^3}\sqrt{x}$, will be the Celerity of the Globe *B* where it falls on the Ground. Then the $\sqrt{x} + \frac{a^3}{b^3}\sqrt{x}$ will be equal to $\sqrt{\frac{a^6}{b^6}x + p - x}$. For $\frac{a^3 + b^3}{b^3}$ write $\frac{r}{s}$, and for $\frac{a^6 - b^6}{b^6}$, $\frac{rt}{ss}$, and that Æquation will become $\frac{r}{s}\sqrt{x} = \sqrt{\frac{rt}{ss}x + p}$, and the Parts being ſquar'd, $\frac{rr}{ss}x = \frac{rt}{ss}x + p$, ſubtract from both Sides $\frac{rt}{ss}x$, and multiply all into ss, and divide by $rr - rt$, and there will ariſe $x = \frac{ssp}{rr - rt}$. Which Æquation would have come out more ſimple, if I had taken $\frac{p}{s}$ for $\frac{a^3 + b^3}{b^3}$, for there would have come out $\frac{ss}{p - t} = x$. Whence making that $p - t$ ſhall be to s as s to x, you'll have x, or ed; to which if you add ec, you'll have dc, and the Point c, in which the Globes ſhall fall upon one another. Q. E. F.

Hitherto I have been ſolving ſeveral Problems. For in learning the Sciences, Examples are of more Uſe than Precepts. Wherefore I have been the larger on this Head. And ſome which occurr'd as I was putting down the reſt, I have given their Solutions without uſing Algebra, that I might ſhew that in ſome Problems that at firſt Sight appear difficult, there is not always Occaſion for Algebra. But now it is Time to ſhew the Solution of Æquations. For after a Problem is brought to an Æquation, you muſt extract the Roots of that Æquation, which are the Quantities that [anſwer or] ſatisfy the Problem.

How ÆQUATIONS *are to be ſolv'd.*

AFTER therefore in the Solution of a Queſtion you are come to an Æquation, and that Æquation is duly reduc'd and order'd; when the Quantities which are ſuppos'd given, are really given in Numbers, thoſe Numbers are to be ſubſtituted in their room in the Æquation, and you'll have a Numeral Æquation, whoſe Root being extracted will ſatisfy the Queſtion. As if in the Diviſion of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the propos'd Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Æquation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Caſe the Radius r is given in Numbers, and the Line q ſubtending the Complement of the given Angle; as if Radius were 10, and the Chord 3; I ſubſtitute thoſe Numbers in the Æquation for r and q, and there comes out the Numeral Æquation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line ſubtending the Complement of the fifth Part of that given Angle.

Of the Nature of the Roots of an Æquation.

But the Root is a Number which being ſubſtituted in the Æquation for the Letter or Species ſignifying the Root, will make all the Terms vaniſh. Thus Unity is the Root of the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, becauſe being writ for x it produces $1 - 1 - 19 + 49 - 30$, that is, nothing. And thus, if for x you write the Number 3, or the Negative Number -5, and in both Caſes there will be produc'd nothing, the Affirmative and Negative Terms in theſe four Caſes deſtroying one another; then ſince any of the Numbers written in the Æquation fulfils the Condition of x, by making all the Terms of the Æquation together equal to nothing, any of them will be the Root of the Æquation.

And that you may not wonder that the ſame Æquation may have ſeveral Roots, you muſt know that there may be more Solutions [than one] of the ſame Problem. As if there was ſought the Interſection of two given Circles; there are two Interſections, and conſequently the Queſtion admits two Anſwers; and then the Æquation determining the

the Interfection will have two Roots, whereby it determines both [Points of] the Interfection, if there be nothing in the Data whereby the Anfwer is determin'd to [only] one Interfection. [*Vide Figure* 87.] And thus, if the Arch APB the fifth Part of AP were to be found, though perhaps you might apply your Thoughts only to the Arch APB, yet the Æquation, whereby the Queftion will be folv'd, will determine the fifth Part of all the Arches which are terminated at the Points A and B; *viz.* the fifth Part of the Arches ASB, $APBSAPB$, $ASBPASB$, and $APBSAPBSAPB$, as well as the fifth Part of the Arch APB; which fifth Part, if you divide the whole Circumference into five equal Parts PQ, QR, RS, ST, TP, will be AT, AQ, ATS, AQR. Wherefore, by feeking the fifth Parts of the Arches which the right Line AB fubtends, to determine all the Cafes the whole Circumference ought to be divided in the five Points P, Q, R, S, T. Wherefore, the Æquation that will determine all the Cafes will have five Roots For the fifth Parts of all thefe Arches depend on the fame Data, and are found by the fame Kind of Calculus; fo that you'll always fall upon the fame Æquation, whether you feek the fifth Part of the Arch APB, or the fifth Part of the Arch ASB, or the fifth Part of any other of the Arches. Whence, if the Æquation by which the fifth Part of the Arch APB is determin'd, fhould not have more than one Root, while by feeking the fifth Part of the Arch ASB we fall upon that fame Æquation; it would follow, that this greater Arch would have the fame fifth Part with the former, which is lefs, becaufe its Subtenfe [or Chord] is exprefs'd by the fame Root of the Æquation. In every Problem therefore it is neceffary, that the Æquation which anfwers fhould have as many Roots as there are different Cafes of the Quantity fought depending on the fame Data, and to be determin'd by the fame Method of Reafoning.

But an Æquation may have as many Roots as it has Dimenfions, and not more. Thus the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, has four Roots, 1, 2, 3, -5, but not more. For any of thefe Numbers writ in the Æquation for x will caufe all the Terms to deftroy one another as we have faid; but befides thefe, there is no Number by whofe Subftitution this will happen. Moreover, the Number and Nature of the Roots will be beft underftood from the Generation of the Æquation. As if we would know how an Æquation is generated, whofe Roots are 1, 2, 3, and -5; we are to ſuppoſe x to ſignify ambiguouſly thoſe Numbers, or x to be $=1$, $x=2$, $x=3$, and $x=-5$, or which is the ſame Thing, $x-1=0$, $x-2=0$, $x-3=0$, and $x+5=0$; and multiplying theſe together, there will come out by the Multiplication of $x-1$ by $x-2$ this Æquation $xx-3x+2=0$, which is of two Dimenſions, and has two Roots 1 and 2. And by the Multiplication of this by $x-3$, there will come out $x^3-6xx+11x-6=0$, an Æquation of three Dimenſions and as many Roots; which again multiply'd by $x+5$ becomes $x^4-x^3-19xx+49x-30=0$, as above. Since therefore this Æquation is generated by four Factors, $x-1$, $x-2$, $x-3$, $x+5$, continually multiply'd by one another, where any of the Factors is nothing, that which is made by all will be nothing; but where none of them is nothing, that which is contain'd under them all cannot be nothing. That is, $x^4-x^3-19xx+49x-30$ cannot be $=0$, as ought to be, except in theſe four Caſes, where $x-1$ is $=0$, or $x-2=0$, or $x-3=0$, or, laſtly, $x+5=0$, and then only the Numbers 1, 2, 3, and -5 can exhibit x, or be the Roots of the Æquation. And you are to reaſon alike of all Æquations. For we may imagine all to be generated by ſuch a Multiplication, although it is uſually very difficult to ſeparate the Factors from one another, and is the ſame Thing as to reſolve the Æquation and extract its Roots. For the Roots being had, the Factors are had alſo.

But the Roots are of two Sorts, Affirmative, as in the Example brought, 1, 2, and 3, and Negative, as -5. And of theſe ſome are often impoſſible. Thus, the two Roots of the Æquation $xx-2ax+bb=0$, which are $a+\sqrt{aa-bb}$ and $a-\sqrt{aa-bb}$, are real when aa is greater than bb; but when aa is leſs than bb, they become impoſſible, becauſe then $aa-bb$ will be a Negative Quantity, and the Square Root of a Negative Quantity is impoſſible. For every poſſible Root, whether it be Affirmative or Negative, if it be multiply'd by it ſelf, produces an Affirmative Square; therefore that will be an impoſſible one which is to produce a Negative Square. By the ſame Argument you may conclude, that the Æquation $x^3-4xx+4x-6=0$, has one real Root, which is 2, and two impoſſible ones $1+\sqrt{-2}$ and $1-\sqrt{-2}$. For any of theſe, 2, $1+\sqrt{-2}$, $1-\sqrt{-2}$ being writ in the Æquation for x, will make all its Terms deſtroy one another; but $1+\sqrt{-2}$, and $1-\sqrt{-2}$, are im-

impossible Numbers, because they suppose the Extraction of the Square Root out of the Negative Number — 2.

But it is just, that the Roots of Æquations should be often impossible, lest they should exhibit the Cases of Problems that are often impossible as if they were possible. As if you were to determine the Intersection of a right Line and a Circle, and you should put two Letters for the Radius of the Circle and the Distance of the right Line from its Center; and when you have the Æquation defining the Intersection, if for the Letter denoting the Distance of the right Line from the Center, you put a Number less than the Radius, the Intersection will be possible; but if it be greater, impossible; and the two Roots of the Æquation, which determine the two Intersections, ought to be either possible or impossible, that they may truly express the Matter. [*Vide Figure* 88.] And thus, if the Circle $CDEF$, and the Ellipsis $ACBF$ cut one another in the Points C, D, E, F, and to any right Line given in Position, as AB, you let fall the Perpendiculars CG, DH, EI, FK, and by seeking the Length of any one of the Perpendiculars, you come at length to an Æquation; that Æquation, where the Circle cuts the Ellipsis in four Points, will have four real Roots, which will be those four Perpendiculars. Now, if the Radius of the Circle, its Center remaining, be diminish'd untill the Points E and F meeting, the Circle at length touches the Ellipse, those two of the Roots which express the Perpendiculars EI and FK now coinciding, will become equal. And if the Circle be yet diminish'd, so that it does not touch the Ellipse in the Point EF, but only cuts it in the other two Points C, D, then out of the four Roots those two which express'd the Perpendiculars EI, FK, which are now become impossible, will become, together with those Perpendiculars, also impossible. And after this Way in all Æquations, by augmenting or diminishing their Terms of the unequal Roots, two will become first equal and then impossible. And thence it is, that the Number of the impossible Roots is always even.

But sometimes the Roots of Æquations are possible, when the Schemes exhibit them as impossible. But this happens by reason of some Limitation in the Scheme, which does not belong to the Æquation. [*Vide Figure* 89.] As if in the Semi-Circle ADB, having given the Diameter AB, and the Chord AD, and having let fall the Perpendicular DC, I was to find the Segment of the Diameter AC, you'll have

have $\frac{ADq}{AB} = AC$. And, by this Æquation, AC is exhibited a real Quantity, where the inscrib'd Line AD is greater than the Diameter AB; but by the Scheme, AC then becomes impossible, *viz.* in the Scheme the Line AD is suppos'd to be inscrib'd in the Circle, and therefore cannot be greater than the Diameter of the Circle; but in the Æquation there is nothing that depends upon that Condition. From this Condition alone of the Lines the Æquation comes out, that AB, AD, and AC are continually proportional. And because the Æquation does not contain all the Conditions of the Scheme, it is not necessary that it should be bound to the Limits of all the Conditions. Whatever is more in the Scheme than in the Æquation may constrain that to Limits, but not this. For which reason, when Æquations are of odd Dimensions, and consequently cannot have all their Roots impossible, the Schemes often set Limits to the Quantities on which all the Roots depend, which 'tis impossible they can exceed, keeping the same Conditions of the Schemes.

Of those Roots that are real ones, the Affirmative and Negative ones lie on contrary Sides, or tend contrary Ways. Thus, in the last Scheme but one, by seeking the Perpendicular CG, you'll light upon an Æquation that has two Affirmative Roots CG and DH, tending from the Points C and D the same Way; and two Negative ones, EI and FK, tending from the Points E and F the opposite Way. Or if in the Line AB there be given any Point P, and the Part of it PG extending from that given Point to some of the Perpendiculars, as CG, be sought, we shall light on an Æquation of four Roots, PG, PH, PI, and PK, whereof the Quantity sought PG, and those that tend from the Point P the same Way with PG, (as PK) will be Affirmative, but those which tend the contrary Way (as PH, PI) Negative.

Where there are none of the Roots of the Æquation impossible, the Number of the Affirmative and Negative Roots may be known from the Signs of the Terms of the Æquation. For there are so many Affirmative Roots as there are Changes of the Signs in a continual Series from $+$ to $-$, and from $-$ to $+$; the rest are Negative. As in the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, where the Signs of the Terms follow one another in this Order, $+ - -$

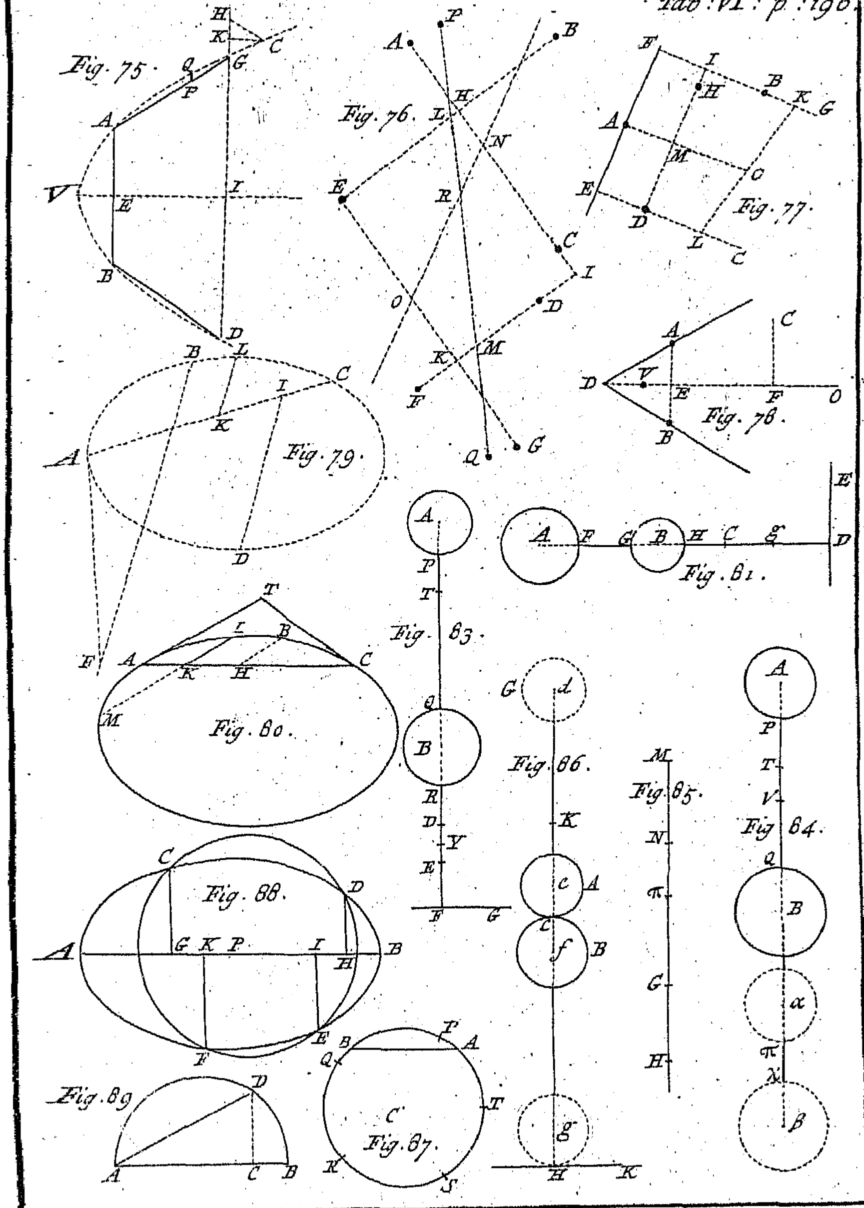
Fig. 75.
Fig. 76.
Fig. 77.
Fig. 78.
Fig. 79.
Fig. 80.
Fig. 81.
Fig. 83.
Fig. 84.
Fig. 85.
Fig. 86.
Fig. 87.
Fig. 88.
Fig. 89

$+-$, the Variations of the second $-$ from the first $+$, of the fourth $+$ from the third $-$, and of the fifth $-$ from the fourth $+$, shew, that there are three Affirmative Roots, and consequently, that the fourth is a Negative one. But where some of the Roots are impossible, the Rule is of no Force, unless as far as those impossible Roots, which are neither Negative nor Affirmative, may be taken for ambiguous ones. Thus in the Æquation $x^3+pxx+3ppx-q=0$, the Signs shew that there is one Affirmative Root and two Negative ones. Suppose $x=2p$, or $x-2p=0$, and multiply the former Æquation by this, $x-2p=0$, and add one Affirmative Root more to the former, and you'll have this Æquation, $x^4-px^3+ppxx\begin{matrix}-2p^3\\-q\end{matrix}x+2pq=0$, which ought to have two Affirmative and two Negative Roots; yet it has, if you regard the Change of the Signs, four Affirmative ones. There are therefore two impossible ones, which for their Ambiguity in the former Case seem to be Negative ones, in the latter, Affirmative ones.

But you may know almost by this Rule how many Roots are impossible. Make a Series of Fractions, whose Denominators are Numbers in this Progression 1, 2, 3, 4, 5, *&c.* going on to the Number which shall be the same as that of the Dimensions of the Æquation; and the Numerators the same Series of Numbers in a contrary Order. Divide each of the latter Fractions by each of the former, and place the Fractions that come out on the middle Terms of the Æquation. And under any of the middle Terms, if its Square, multiply'd into the Fraction standing over its Head, is greater than the Rectangle of the Terms on both Sides, place the Sign $+$, but if it be less, the Sign $-$; but under the first and last Term place the Sign $+$. And there will be as many impossible Roots as there are Changes in the Series of the underwritten Signs from $+$ to $-$, and $-$ to $+$. As if you have the Æquation $x^3+pxx+3ppx-q=0$; I divide the second of the Fractions of this Series $\frac{3}{1}.\frac{2}{2}.\frac{1}{3}$, *viz.* $\frac{2}{2}$ by the first $\frac{3}{1}$, and the third $\frac{1}{3}$ by the second $\frac{2}{2}$, and I place the Fractions that come out, *viz.* $\frac{1}{3}$ and $\frac{1}{3}$ upon the mean Terms of the Æquation, as follows;

$$\begin{array}{ccccccc} & & \frac{1}{3} & & \frac{1}{3} & & \\ x^3 & + & pxx & + & 3ppx & - & q=0. \\ + & & - & & + & & + \end{array}$$

Then

Then, becauſe the Square of the ſecond Term pxx multiply'd into the Fraction over its Head $\frac{1}{3}$, *viz.* $\frac{ppx^4}{3}$ is leſs than $3ppx^4$, the Rectangle of the firſt Term x^3 and third $3ppx$, I place the Sign $-$ under the Term pxx. But becauſe $9p^4xx$ (the Square of the third Term $3ppx$) multiply'd into the Fraction over its Head $\frac{1}{3}$, is greater than nothing and therefore much greater than the Negative Rectangle of the ſecond Term pxx, and the fourth $-q$, I place the Sign $+$ under that third Term. Then, under the firſt Term x^3 and the laſt $-q$, I place the Sign $+$. And the two Changes of the underwritten Signs; which are in this Series $+ - + +$, the one from $+$ into $-$, and the other from $-$ into $+$, ſhew that there are two impoſſible Roots. And thus the Æquation $x^3 - 4xx + 4x - 6 = 0$ has two impoſſible Roots,

$$\begin{array}{cccccc} & \frac{1}{3} & \frac{1}{3} & & \\ x^3 & -4xx & +4x & -6 & =0. \\ + & + & - & + & \end{array}$$

Alſo the Æquation $x^4 - 6xx - 3x - 2 = 0$ has two,

$$\begin{array}{cccccc} & \frac{3}{8} & \frac{4}{9} & \frac{3}{8} & & \\ x^4 & * & -6xx & -3x & -2 & =0. \\ + & + & + & - & + & \end{array}$$

For this Series of Fractions $\frac{1}{4} . \frac{2}{3} . \frac{4}{1}$, by dividing the ſecond by the firſt, and the third by the ſecond, and the fourth by the third, gives this Series $\frac{3}{8} . \frac{4}{9} . \frac{3}{8}$, to be placed upon the middle Terms of the Æquation. Then the Square of the ſecond Term, which is here nothing, multiply'd into the Fraction over Head, *viz.* $\frac{3}{8}$, produces nothing, which is yet greater than the Negative Rectangle $-6x^6$ contain'd under the Terms x^4 and $-6xx$. Wherefore, under the Term that is wanting I write $+$. In the reſt I go on as in the former Example; and there comes out this Series of the underwritten Signs $+ + + - +$, where two Changes ſhew there are two impoſſible Roots. And after the ſame Way in the Æquation $x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 = 0$, are diſcover'd two impoſſible Roots, as follows;

$$\begin{array}{ccccccc} & \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{5} & & \\ x^5 & -4x^4 & +4x^3 & -2xx & -5x & -4 & =0. \\ + & + & - & + & + & + & \end{array}$$

Where two or more Terms are at once wanting, under the firſt of the deficient Terms you muſt write the Sign $-$, under the ſecond the Sign $+$, under the third the Sign $-$, and ſo on, always varying the Signs, except that under the laſt

last of the deficient Terms you must always place $+$, where the Terms next on both Sides the deficient Terms have contrary Signs. As in the Æquations $\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & a^5 = 0, \\ + & + & - & + & - & + \end{array}$ and $\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & -a^5 = 0 \\ + & + & - & + & + & + \end{array}$; the first whereof has four, and the latter two impossible Roots. Thus also the Æquation,

$$\begin{array}{cccccccc} x^7 & -\overset{\frac{3}{7}}{2x^6} & +\overset{\frac{5}{9}}{3x^5} & -\overset{\frac{3}{5}}{2x^4} & +\overset{\frac{3}{5}}{x^3} & \overset{\frac{5}{9}}{*} & \overset{\frac{3}{7}}{*} & -3=0 \\ + & - & + & - & + & - & + & + \end{array}$$

has six impossible Roots.

Hence also may be known whether the impossible Roots are among the Affirmative or Negative ones. For the Signs of the Terms over Head of the subscrib'd *changing* Terms shew, that there as many impossible Affirmative [Roots] as there are Variations of them, and as many Negative ones *as there are Successions without Variations. Thus, in the* Æquation $\begin{array}{cccccc} x^5 & -4x^4 & +4x^3 & -2xx & -5x & -4=0 \\ + & + & - & + & + & + \end{array}$, because by the Signs that are writ underneath that are changeable, *viz.* $+ - +$, by which it is shewn there are two impossible Roots, the Terms over Head $-4x^4 + 4x^3 - 2xx$ have the Signs $- + -$, which by two Variations shew there are two Affirmative Roots; therefore there will be two impossible Roots among the Affirmative ones. Since therefore the Signs of all the Terms of the Æquation $+ - + - - -$, by three Variations shew that there are three Affirmative Roots, and that the other two are Negative, and that among the Affirmative ones there are two impossible ones; it follows that there are, *viz.* one true affirmative Root, two negative ones, and two impossible ones. Now, if the Æquation had been $\begin{array}{cccccc} x^5 & -4x^4 & -4x^3 & -2xx & -5x & -4=0 \\ + & + & - & - & + & + \end{array}$, then the Terms over Head of the subscrib'd former Terms $+ -$, *viz.* $-4x^4 - 4x^3$, by their Signs that don't change $-$ and $-$, shew, that one of the Negative Roots is impossible; and the Terms over the former underwritten varying Terms $- +$, *viz.* $-2xx - 5x$, by their Terms not varying, $-$ and $-$, shew that one of the Negative Roots are impossible. Wherefore, since the Signs of the Æquation $+ - - - - -$ by one Variation shew there is one Affirmative Root, and that the other four are Negative;

Negative, it follows, there is one Affirmative, two Negative, and two Impossible ones. And this is so where there are not more impossible Roots than what are discover'd by the Rule preceding. For there may be more, although it seldom happens.

Moreover, all the Affirmative Roots of any Æquation may be chang'd into Negative ones, and the Negative into Affirmative ones, and that only by changing of the alternate Terms [*i. e.* every other Term]. Thus, in the Æquation $x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 = 0$, the three Affirmative Roots will be chang'd into Negative ones, and the two Negative ones into Affirmatives, by changing only the Signs of the second, fourth, and sixth Terms, as is done here, $x^5 + 4x^4 + 4x^3 + 2xx - 5x + 4 = 0$. This Æquation has the same Roots with the former, unless that in this, those Roots are Affirmative that were there Negative, and Negative here that there were Affirmative; and the two impossible Roots, which lay hid there among the Affirmative ones, lie hid here among the Negative ones; so that these being subduc'd, there remains only one Root truly Negative.

Of the Transmutations of Æquations.

There are also other Transmutations of Æquations which are of Use in divers Cases. For we may suppose the Root of an Æquation to be compos'd any how out of a known and an unknown Quantity, and then substitute what we suppose equivalent to it. As if we suppose the Root to be equal to the Sum or Difference of any known and unknown Quantity. For, after this Rate, we may augment or diminish the Roots of the Æquation by that known Quantity, or subtract them from it; and thereby cause that some of them that were before Negative shall now become Affirmative, or some of the Affirmative ones become Negative, or also that all shall become Affirmative or all Negative. Thus, in the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, if I have a mind to augment the Roots by Unity, I suppose $x + 1 = y$, or $x = y - 1$; and then for x I write $y - 1$ in the Æquation, and for the Square, Cube, or Biquadrate of x, I write the like Power of $y - 1$, as follows;

x^4

$$\begin{array}{r|rrrrr}
x^4 & y^4 - 4y^3 & + \; 6yy & - \; 4y & + 1 \\
-x^3 & -y^3 & + \; 3yy & - \; 3y & + 1 \\
-19xx & & -19yy & +38y & -19 \\
+49x & & & +49y & -49 \\
-30 & & & & -30 \\
\hline
\text{Sum} & y^4 - 5y^3 & -10yy & +80y & -96 = 0.
\end{array}$$

And the Roots of the Æquation that is produc'd, (*viz.*) $y^4 - 5y^3 - 10yy + 80y - 96 = 0$, will be $2, 3, 4, -4$, which before were $1, 2, 3, -5$, *i. e.* bigger by Unity. Now, if for x I had writ $y + 1\frac{1}{2}$, there would have come out the Æquation $y^4 + 5y^3 - 10yy - \frac{5}{4}y + \frac{39}{16} = 0$, whereof there be two Affirmative Roots, $\frac{1}{2}$ and $1\frac{1}{2}$, and two Negative ones, $-\frac{1}{2}$ and $-6\frac{1}{2}$. But by writing $y - 6$ for x, there would have come out an Æquation whose Roots would have been $7, 8, 9, 1$, *viz.* all Affirmative; and writing for the same [x] $y + 4$, there would have come out those Roots diminish'd by 4, *viz.* $-3 - 2 - 1 - 9$, all of them Negative.

And after this Way, by augmenting or diminishing the Roots, if any of them are impossible, they will sometimes be more easily detected this Way than before.

Thus, in the Æquation $x^3 - 3aax - 3a^3 = 0$, there are no Roots that appear impossible by the preceding Rule; but if you augment the Roots by the Quantity a, writing $y - a$ for x, you may by that Rule discover two impossible Roots in the Æquation resulting, $y^3 - 3ayy - a^3 = 0$.

By the same Operation you may also take away the second Terms of Æquations; which will be done, if you subduct the known Quantity [or Co-efficient] of the second Term of the Æquation propos'd, divided by the Number of Dimensions [of the highest Term] of the Æquation, from the Quantity which you assume to signify the Root of the new Æquation, and substitute the Remainder for the Root of the Æquation propos'd. As if there was propos'd the Æquation $x^3 - 4xx + 4x - 6 = 0$, I subtract the known Quantity [or Co-efficient] of the second Term, which is -4, divided by the Number of the Dimensions of the Æquation, *viz.* 3, from the Species [or Letter] which is assum'd to signify the new Root, suppose from y, and the Remainder $y + \frac{4}{3}$ I substitute for x, and there comes out

$$\begin{array}{r} y^3 + 4yy + \frac{16}{3}y + \frac{64}{27} \\ -4yy - \frac{32}{3}y - \frac{64}{9} \\ +4y + \frac{16}{3} \\ -6 \\ \hline y^3 \quad * \quad -\frac{4}{3}y - \frac{146}{27} = 0. \end{array}$$

By the same Method, the third Term of an Æquation may be alſo taken away. Let there be propos'd the Æquation $x^4 - 3x^3 + 3xx - 5x - 2 = 0$, and make $x = y - e$, and ſubſtituting $y - e$ in the room of x, there will ariſe this Æquation;

$$y^4 \begin{matrix} -4e \\ -3 \end{matrix} y^3 \begin{matrix} +6ee \\ +9e \\ +3 \end{matrix} yy \begin{matrix} -4e^3 \\ -9ee \\ -6e \\ -5 \end{matrix} y \left.\begin{matrix} +e^4 \\ +3e^3 \\ +3ee \\ +5e \\ -2 \end{matrix}\right\} = 0.$$

The third Term of this Æquation is $6ee + 9e + 3$ multiply'd by yy. Where, if $6ee + 9e + 3$ were nothing, you'd have what you deſir'd. Let us ſuppoſe it therefore to be nothing, that we may thence find what Number ought to be ſubſtituted in this Caſe for e, and we ſhall have the Quadratick Æquation $6ee + 9e + 3 = 0$, which divided by 6 will become $ee + \frac{3}{2}e + \frac{1}{2} = 0$, or $ee = -\frac{3}{2}e - \frac{1}{2}$, and extracting the Root $e = -\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}$, or $= -\frac{3}{4} \pm \sqrt{\frac{1}{16}}$, that is, $= -\frac{3}{4} \pm \frac{1}{4}$, and conſequently equal $\left\{\begin{matrix} -\frac{1}{2} \\ -1 \end{matrix}\right\}$ Whence $y - e$ will be either $y + \frac{1}{2}$, or $y + 1$. Wherefore, ſince $y - e$ was writ for x; in the room of $y - e$ there ought to be writ $y + \frac{1}{2}$, or $y + 1$ for x, that the third Term of the Æquation that reſults may be taken away. And that will happen in both Caſes. For if for x you write $y + \frac{1}{2}$, there will ariſe this Æquation, $y^4 - y^3 - \frac{15}{4}y - \frac{77}{16} = 0$; but if you write $y + 1$, there will ariſe this Æquation, $y^4 + y^3 - 4y - 12 = 0$.

Moreover, the Roots of Æquations may be multiply'd or divided by given Numbers; and after this Rate, the Terms of Æquations be diminiſh'd, and Fractions and Radical Quantities ſometimes be taken away. As if the Æquation were $y^3 - \frac{4}{3}y - \frac{146}{27} = 0$; in order to take away the Fractions, I ſuppoſe y to be $= \frac{1}{3}z$, and then by ſubſtituting $\frac{1}{3}z$ for

for y, there comes out this new Æquation, $\frac{z^3}{27} - \frac{12z}{27} - \frac{146}{27} = 0$, and having rejected the common Denominator of the Terms, $z^3 - 12z - 146 = 0$, the Roots of which Æquation are thrice greater than before. And again, to diminish the Terms of this Æquation, if you write $2v$ for z, there will come out $8v^3 - 24v - 146 = 0$, and dividing all by 8, you'll have $v^3 - 3v - 18\frac{1}{4} = 0$; the Roots of which Æquation are half of the Roots of the former. And here, if at last you find v make $2v = z$, $\frac{1}{3}z = y$, and $y + \frac{4}{3} = x$, and you'll have x the Root of the Æquation as first propos'd.

And thus, in the Æquation $x^3 - 2x + \sqrt{3} = 0$, to take away the Radical Quantity $\sqrt{3}$; for x I write $y\sqrt{3}$, and there comes out the Æquation $3y^3\sqrt{3} - 2y\sqrt{3} + \sqrt{3} = 0$, which, dividing all the Terms by the $\sqrt{3}$, becomes $3y^3 - 2y + 1 = 0$.

Again, the Roots of an Æquation may be chang'd into their Reciprocals, and after this Way the Æquation may be sometimes reduc'd to a more commodious Form. Thus, the last Æquation $3y^3 - 2y + 1 = 0$, by writing $\frac{1}{z}$ for y, becomes $\frac{3}{z^3} - \frac{2}{z} + 1 = 0$, and all the Terms being multiply'd by z^3, and the Order of the Terms chang'd, $z^3 - 2zz + 3 = 0$. The last Term but one of an Æquation may also by this Method be taken away, as the second was taken away before, as you see done in the precedent Æquation; or if you would take away the last but two, it may be done as you have taken away the third. Moreover, the least Root may thus be converted into the greatest, and the greatest into the least, which may be of some Use in what follows. Thus, in the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, whose Roots are 3, 2, 1, -5, if you write $\frac{1}{y}$ for x, there will come out the Æquation $\frac{1}{y^4} - \frac{1}{y^3} - \frac{19}{yy} + \frac{49}{y} - 30 = 0$, which, multiplying all the Terms by y^4, and dividing them by 30, the Signs being chang'd, becomes $y^4 - \frac{49}{30}y^3 + \frac{19}{30}y^2 + \frac{1}{30}y - \frac{1}{30} = 0$, the Roots whereof

are

are $\frac{1}{3}$, $\frac{1}{2}$, 1, $-\frac{1}{5}$; the greatest of the Affirmative Roots 3 being now changed into the least $\frac{1}{3}$, and the least 1 being now made greatest, and the Negative Root -5, which of all was the most remote from 0, now coming nearest to it.

There are also other Transmutations of Æquations, but which may all be perform'd after that Way of transmutating we have shewn, when we took away the third Term of the Æquation.

From the Generation of Æquations it is evident, that the known Quantity of the second Term of the Æquation, if its Sign be chang'd, is equal to the Aggregate [or Sum] of all the Roots [added together] under their proper Signs; and that of the third Term equal to the Aggregate of the Rectangles of each two of the Roots; that of the fourth, if its Sign be chang'd, is equal to the Aggregate of the Contents under each three of the Roots; that of the fifth is equal to the Aggregate of the Contents under each four, and so on *ad infinitum*. Let us assume $x=a$, $x=b$, $x=-c$, $x=d$, &c. or $x-a=0$, $x-b=0$, $x+c=0$, $x-d=0$, and by the continual Multiplication of these we may generate Æquations as above. Now, by multiplying $x-a$ by $x-b$ there will be produc'd the Æquation $xx \begin{array}{l}-a\\-b\end{array} x + ab = 0$; where the known Quantity of the second Term, if its Signs are chang'd, *viz.* $a+b$, is the Sum of the two Roots a and b, and the known Quantity of the third Term is the only Rectangle contain'd under both. Again, by multiplying this Æquation by $x+c$, there will be produc'd the Cubick Æquation $x^3 \begin{array}{l}-a\\-b\\+c\end{array} xx \begin{array}{l}+ab\\-ac\\-bc\end{array} x + abc = 0$, where the known Quantity of the second Term having its Signs chang'd, *viz.* $a+b-c$, is the Sum of the Roots a, and b, and $-e$; the known Quantity of the third Term $ab-ac-bc$ is the Sum of the Rectangles under each two of the Terms a and b, a and $-c$, b and $-c$; and the known Quantity of the fourth Term under its Sign chang'd, $-abc$, is the only Content generated by the continual Multiplication of all the Terms, a by b into $-c$. Moreover, by multiplying that Cubick Æquation by $x-d$, there will be produc'd this Biquadratick one;

x^4

$$x^4 \begin{matrix}-a\\-b\\+c\\-d\end{matrix} \; x^3 \begin{matrix}+ab\\-ac\\-bc\\+ad\\+bd\\-cd\end{matrix} \; xx \begin{matrix}+abc\\-abd\\+bcd\\+acd\end{matrix} \; x - abcd = 0.$$

Where the known Quantity of the fecond Term under its Signs chang'd, *viz.* $a+b-c+d$, is the Sum of all the Roots; that of the third, $ab-ac-bc+ad+bd-cd$, is the Sum of the Rectangles under every two; that of the fourth, its Signs being chang'd, $-abc+abd-bcd-acd$, is the Sum of the Contents under each Ternary; that of the fifth, $-abcd$, is the only Content under them all. And hence we firft infer, that all the rational Roots of any Æquation that involves neither Surds nor Fractions, and the Rectangles of any two of the Roots, or the Contents of any three or more of them, are fome of the Integral Divifors of the laft Term; and therefore when it is evident, that there is no Divifor of the laft Term, or Root of the Æquation, or Rectangle, or Content of two or more, it will alfo be evident that there is no Root, or Rectangle, or Content of Roots, except what is Surd.

Let us fuppofe now, that the known Quantities of the Terms of [any] Æquation under their Signs chang'd, are p, q, r, s, t, v, &c. *viz.* that of the fecond p, that of the third q, of the fourth r, of the fifth s, and fo on. And the Signs of the Terms being rightly obferv'd, make $p=a$, $pa+2q=b$, $pb+qa+3r=c$, $pc+qb+ra+4s=d$, $pd+qc+rb+sa+5t=e$, $pe+qd+rc+sb+ta+6v=f$, and fo on *ad infinitum*, obferving the Series of the Progreffion. And a will be the Sum of the Roots, b the Sum of the Squares of each of the Roots, c the Sum of the Cubes, d the Sum of the Biquadrates, e the Sum of the Quadrato-Cubes, f the Sum of the Cubo-Cubes [or fixth Power] and fo on. As in the Æquation $x^4-x^3-19xx+49x-30=0$, where the known Quantity of the fecond Term is -1, of the third -19, of the fourth $+49$, of the fifth -30; you muft make $1=p$, $19=q$, $-49=r$, $30=s$. And there will thence arife $a=(p=)\ 1$, $b=(pa+2q=1+38=)\ 39$, $c=(pb+qa+3r=39\ 19-147=)-89$, $d=(pc+qb+ra+4s=-89++741-49+120=)\ 723$. Wherefore the Sum of the

Root

Roots will be 1, the Sum of the Squares of the Roots 39, the Sum of the Cubes -89, and the Sum of the Biquadrates 723, *viz.* the Roots of that Æquation are 1, 2, 3, and -5, and the Sum of these $1+2+3-5$ is 1; the Sum of the Squares, $1+4+9+25$, is 39; the Sum of the Cubes, $1+8+27-125$, is -89; and the Sum of the Biquadrates, $1+16+81+625$, is 723.

And hence are collected the Limits between which the Roots of the Æquation shall consist, if none of them is impossible. For when the Squares of all the Roots are Affirmative, the Sum of the Squares will be Affirmative, and therefore greater than the Square of the greatest Root. And by the same Argument, the Sum of the Biquadrates of all the Roots will be greater than the Biquadrate of the greatest Root, and the Sum of the Cubo-Cubes greater than the Cubo-Cube of the greatest Root. Wherefore, if you desire the Limit which no Roots can pass, seek the Sum of the Squares of the Roots, and extract its Square Root. For this Root will be greater than the greatest Root of the Æquation. But you'll come nearer the greatest Root if you seek the Sum of the Biquadrates, and extract its Biquadratick Root; and yet nearer, if you seek the Sum of the Cubo-Cubes, and extract its Cubo-Cubical Root, and so on *ad infinitum.*

Of the Limits of Æquations.

Thus, in the precedent Æquation, the Square Root of the Sum of the Squares of the Roots, or $\sqrt{39}$, is $6\frac{1}{2}$ nearly, and $6\frac{1}{2}$ is farther distant from 0 than any of the Roots 1, 2, 3, -5. But the Biquadratick Root of the Sum of the Biquadrates of the Roots, *viz.* $\sqrt[4]{723}$, which is $5\frac{1}{4}$ nearly, comes nearer to the Root that is most remote from nothing, *viz.* -5.

If, between the Sum of the Squares and the Sum of the Biquadrates of the Roots you find a mean Proportional, that will be a little greater than the Sum of the Cubes of the Roots connected under Affirmative Signs. And hence, the half Sum of this mean Proportional, and of the Sum of the Cubes collected under their proper Signs, found as before, will be greater than the Sum of the Cubes of the Affirmative Roots, and the half Difference greater than the Sum of the Cubes of the Negative Roots. And consequently, the greatest of the Affirmative Roots will be less than the Cube Root of that Semi-Difference. Thus, in the precedent Æquation, a mean Proportional between the Sum of the Squares

of

of the Roots 39, and the Sum of the Biquadrates 723, is nearly 168. The Sum of the Cubes under their proper Signs was, as above, -89, the half Sum of this and 168 is $39\frac{1}{2}$, the Semi-Difference $128\frac{1}{2}$. The Cube Root of the former which is about $3\frac{1}{2}$, is greater than the greatest of the Affirmative Roots 3. The Cube Root of the latter, which is $5\frac{1}{21}$ nearly, is greater than the Negative Root -5. By which Example it may be seen how near you may come this Way to the Root, where there is only one Negative Root or one Affirmative one. And yet you might come nearer yet, if you found a mean Proportional between the Sum of the Biquadrates of the Roots and the Sum of the Cubo-Cubes, and if from the Semi-Sum and Semi-Difference of this, and of the Sum of the Quadrato-Cube of the Roots, you extracted the Quadrato-Cubical Roots. For the Quadrato-Cubical Root of the Semi-Sum would be greater than the greatest Affirmative Root, and the Quadrato-Cubick Root of the Semi-Difference would be greater than the greatest Negative Root, but by a less Excess than before. Since therefore any Root, by augmenting and diminishing all the Roots, may be made the least, and then the least converted into the greatest, and afterwards all besides the greatest be made Negative, it is manifest how [any] Root desired may be found nearly.

If all the Roots except two are Negative, those two may be both together found this Way. The Sum of the Cubes of those two Roots being found according to the precedent Method, as also the Sum of the Quadrato-Cubes, and the Sum of the Quadrato-Quadrato-Cubes of all the Roots; between the two latter Sums seek a mean Proportional, and that will be the Difference between the Sum of the Cubo-Cubes of the Affirmative Roots, and the Sum of the Cubo-Cubes of the Negative Roots nearly; and consequently, the half Sum of this mean Proportional, and of the Sum of the Cubo-Cubes of all the Roots, will be the Semi-Sum of the Cubo-Cubes of the Affirmative Roots, and the Semi-Difference will be the Sum of the Cubo-Cubes of the Negative Roots. Having therefore both the Sum of the Cubes, and also the Sum of the Cubo-Cubes of the two Affirmative Roots, from the double of the latter Sum subtract the Square of the former Sum, and the Square Root of the Remainder will be the Difference of the Cubes of the two Roots. And having both the Sum and Difference of the Cubes, you'll have the Cubes themselves. Extract their Cube Roots, and

you'll

you'll nearly have the two Affirmative Roots of the Æquation. And if in higher Powers you should do the like, you'll have the Roots yet more nearly. But these Limitations, by reason of the Difficulty of the Calculus, are of less Use, and extend only to those Æquations that have no imaginary Roots, wherefore I will now shew how to find the Limits another Way, which is more easy, and extends to all Æquations.

Multiply every Term of the Æquation by the Number of its Dimensions, and divide the Product by the Root of the Æquation; then again multiply every one of the Terms that come out by a Number less by Unity than before, and divide the Product by the Root of the Æquation, and so go on, by always multiplying by Numbers less by Unity than before, and dividing the Product by the Root, till at length all the Terms are destroy'd, whose Signs are different from the Sign of the first or highest Term, except the last; and that Number will be greater than any Affirmative Root; which being writ in the Terms that come out for [or in room of] the Root, makes the Aggregate of those which were each Time produc'd by Multiplication to have always the same Sign with the first or highest Term of the Æquation. As if there was propos'd the Æquation $x^5\ 2x^4 - 10x^3 + 30xx + 63x - 120 = 0$. I first multiply this thus;

$$\begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 & 0 \\ x^5 & -2x^4 & -10x^3 & +30xx & +63x & -120. \end{array}$$

Then I again multiply the Terms that come out divided by x, thus;

$$\begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \\ 5x^4 & -8x^3 & -30xx & +60x & +63. \end{array}$$

And dividing the Terms that come out again by x, there comes out $20x^3 - 24xx - 60x + 60$; which, to lessen them, I divide by the greatest common Divisor 4, and you have $5x^3 - 6xx - 15x + 15$. These being again multiply'd by the Progression 3, 2, 1, 0, and divided by x, becomes $5xx - 4x - 5$. And these multiply'd by the Progression 2, 1, 0, and divided by $2x$ become $5x - 2$. Now, since the highest Term of the Æquation x^5 is Affirmative, I try what Number writ in these Products for x will cause them all to be Affirmative. And by trying 1, you have $5x - 2 = 3$ Affirmative; but $5xx - 4x - 5$, you have -4 Negative. Wherefore the Limit will be greater than 1. I therefore try some greater Number, as 2; and substituting 2 in each for x, they become

$$5x - 2 = 8$$
$$5xx - 4x - 5 = 7$$
$$5x^3 - 6xx - 15x + 15 = 1$$
$$5x^4 - 8x^3 - 30xx + 60x + 63 = 79$$
$$x^5 - 2x^4 - 10x^3 + 30xx + 63x - 120 = 46.$$

Wherefore, since the Numbers that come out 8 . 7 . 1 . 79 . 46 . are all Affirmative, the Number 2 will be greater than the greatest of the Affirmative Roots. In like manner, if I would find the Limit of the Negative Roots, I try Negative Numbers. Or that which is all one, I change the Signs of every other Term, and try Affirmative ones. But having chang'd the Signs of every other Term, the Quantities in which the Numbers are to be substituted, will become

$$5x + 2$$
$$5xx + 4x - 5$$
$$5x^3 + 6xx - 15x - 15$$
$$5x^4 + 8x^3 - 30xx - 60x + 63$$
$$x^5 + 2x^4 - 10x^3 - 30xx + 63x + 120.$$

Out of these I chuse some Quantity wherein the Negative Terms seem most prevalent; suppose $5x^4 + 8x - 30xx - 60x + 63$, and here substituting for x the Numbers 1 and 2, there come out the Negative Numbers -14 and -33. Whence the Limit will be greater than -2. But substituting the Number 3, there comes out the Affirmative Number 234. And in like manner in the other Quantities, by substituting the Number 3 there comes out always an Affirmative Number, which may be seen by bare Inspection. Wherefore the Number -3 is greater than all the Negative Roots. And so you have the Limits 2 and -3, between which are all the Roots.

But the Invention of Limits is of Use both in the Reduction of Æquations by Rational Roots, and in the Extraction of Surd Roots out of them; least we might sometimes go about to look for the Root beyond these Limits. Thus, in the last Æquation, if I would find the Rational Roots, if perhaps it has any; from what we have said, it is certain they can be no other than the Divisors of the last Term of the Æquation, which here is 120. Then trying all its Divisors, if none of them writ in the Æquation for x would make all the Terms vanish, it is certain that the Æquation will admit of no Root but what is Surd. But there are many Divisors of the last Term 120, *viz.* 1. — 1. 2. — 2. 3. — 3. 4. — 4. 5. — 5. 6 — 6. 8. — 8. 10. — 10. 12. — 12. 15. — 15. 20. — 20. 24. — 24. 30. — 30. 40. — 40. 60. — 60. 120. and — 120. To try all these Divisors would be tedious. But it being known that the Roots are between 2 and — 3, we are free'd from that Labour. For now there will be no need to try the Divisors, unless those only that are within these Limits, *viz.* the Divisors 1, and — 1. and — 2. For if none of these are the Root, it is certain that the Æquation has no Root but what is Surd.

Hitherto I have treated of the Reduction of Æquations which admit of Rational Divisors; but before we can conclude, that an Æquation of four, six, or more Dimensions is irreducible, we must first try whether or not it may be reduc'd by any Surd Divisor; or, which is the same Thing, you must try whether the Æquation can be so divided into two equal Parts, that you can extract the Root out of both. But that may be done by the following Method.

The Reduction of Æquations by Surd Divisors.

Dispose the Æquation according to the Dimension of some certain Letter, so that all its Terms jointly under their proper Signs, may be equal to nothing, and let the highest Term be affected with an Affirmative Sign. Then, if the Æquation be a Quadratick, (for we may add this Case for the Analogy of the Matter) take from both Sides the lowest Term, and add one fourth Part of the Square of the known Quantity of the *middle* Term. As if the Æquation be $xx - ax - b = 0$, subtract from both Sides $-b$, and add $\frac{1}{4}aa$, and there will come out $xx - ax + \frac{1}{4}aa = b + \frac{1}{4}aa$, and extracting on both Sides the Root, you'll have $x - \frac{1}{2}a = \pm\sqrt{b + \frac{1}{4}aa}$, or $x = \frac{1}{2}a \pm \sqrt{b + \frac{1}{4}aa}$.

Now, if the Æquation be of four Dimensions, suppose $x^4 + px^3 + qxx + rx + s = 0$, where p, q, r, and s denote the known Quantities of the Terms of the Æquation affected by their proper Signs, make

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}\alpha = \beta.$$
$$s - \tfrac{1}{4}\alpha\alpha = \zeta.$$

Then put for n some common Integral Divisor of the Terms β and 2ζ, that is not a Square, and which ought to be odd, and divided by 4 to leave Unity, if either of the Terms p and r be odd. Put also for k some Divisor of the

Quantity

Quantity $\frac{\beta}{n}$ if p be even; or half of the odd Divisor, if p be odd; or nothing, if the Dividual β be nothing. Take the Quotient from $\frac{1}{2}pk$, and call the half of the Remainder l. Then for Q put $\frac{\alpha + nkk}{2}$, and try if n divides $QQ - s$, and the Root of the Quotient be rational and equal to l; which if it happen, add to each Part of the Æquation $nkkxx + 2nklx + nll$, and extract the Root on both Sides, there coming out $xx + \frac{1}{2}px + Q = \sqrt{n}$ into $\overline{kx + l}$.

For Example, let there be propos'd the Æquation $x^4 + 12x - 17 = 0$, and because p and q are both here wanting, and r is 12, and l is -17, having substituted these Numbers, you'll have $\alpha = 0$, $\beta = 12$, and $\zeta = -17$, and the only common Divisor of β and 2ζ, *viz.* 2, will be n. Moreover, $\frac{\beta}{n}$ is 6, and its Divisors 1, 2, 3, and 6, are successively to be try'd for k, and -3, -2, -1, $-\frac{1}{2}$, for l respectively. But $\frac{\alpha + nkk}{2}$, that is, kk is equal to Q. Moreover, $\sqrt{\frac{QQ - s}{n}}$, that is, $\sqrt{\frac{QQ + 17}{2}}$ is equal to l. Where the even Numbers 2 and 6 are writ for k, Q is 4 and 36, and $QQ - s$ will be an odd Number, and consequently cannot be divided by n or 2. Wherefore those Numbers 2 and 6 are to be rejected. But when 1 and 3 are writ for k, Q is 1 and 9, and $QQ - s$ is 18 and 98, which Numbers may be divided by n, and the Roots of the Quotients extracted. For they are ± 3 and ± 7; whereof only -3 agrees with l. I put therefore $k = 1$, $l = -3$, and $Q = 1$, and I add the Quantity $nkkxx + 2nklx + nll$, that is, $2xx - 12x + 18$ to each Part of the Æquation, and there comes out $x^4 + 2xx + 1 = 2xx - 12x + 18$, and extracting on both Sides the Root $xx + 1 = x\sqrt{2} - 3\sqrt{2}$. But if you had rather avoid the Extraction of the Root, make $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, and you'll find, as before, $xx + 1 = \pm\sqrt{2} \times \overline{x - 3}$. And if again you extract the Root of this Æquation, there will come out $x = \pm\frac{1}{2}\sqrt{2} \pm \sqrt{\frac{-1}{2} \pm 3\sqrt{2}}$, that is, according to the

Variation of the Signs $x = \frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}$, and $x = \frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}}$. Alſo $x = -\frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}$, and $x = -\frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}}$. Which are four Roots of the Æquation at firſt propos'd, $x^4 + 12x - 17 = 0$. But the two laſt of them are impoſſible.

Let us now propoſe the Æquation $x^4 - 6x^3 - 58xx - 114x - 11 = 0$, and by writing -6, -58, -114, and -11, for p, q, r, and s reſpectively, there will ariſe $-67 = \alpha$, $-315 = \beta$, and $-1133\frac{1}{4} = \zeta$; the only common Diviſor of the Numbers β and 2ζ, or of -315 and $-\frac{4533}{2}$ is 3, and conſequently will be here n, and the Diviſors of $\frac{\beta}{n}$ or -105, are 3, 5, 7, 15, 21, 35, and 105, which are therefore to be try'd for k. Wherefore, I try firſt 3, and the Quotient -35 which (comes out by dividing $\frac{\beta}{n}$ by k, or -105 by 3) I ſubtract from $\frac{1}{2}pk$, or -3×3, and there remains 26, the half whereof, 13, ought to be l. But $\frac{\alpha + nkk}{2}$, or $\frac{-67 + 27}{2}$, that is, -20, will be Q, and $QQ - s$ will be 411, which may be divided by n, or 3, but the Root of the Quotient, 137, cannot be extracted. Wherefore I reject 3, and try 5 for k. The Quotient that now comes out by dividing $\frac{\beta}{n}$ by k, or -105 by 5, is -21; and ſubtracting this from $\frac{1}{2}pk$, or -3×5, there remains 6, the half whereof, 3, is l. Alſo Q, or $\frac{\alpha + nkk}{2}$, that is, $\frac{-67 + 75}{2}$, is the Number 4. And $QQ - s$, or $16 + 11$, may be divided by n; and the Root of the Quotient, which is 9, being extracted, *i. e.* 3 agrees with l. Wherefore I conclude, that l is $= 3$, $k = 5$, $Q = 4$, and $n = 3$; and if $nkkxx + 2nklx + nll$, that is, $75xx + 90x + 27$, be added to each Part of the Æquation, the Root may be extracted on both Sides, and there will come out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, or $xx - 3x + 4 = \pm\sqrt{3} \times$

$\overline{5x+3}$; and the Root being again extracted, $x = \frac{3 \pm 5\sqrt{3}}{2} \pm \sqrt{17 \pm \frac{21 \times \sqrt{3}}{2}}$.

Thus, if there was propos'd the Æquation $x^4 - 9x^3 + 15xx - 27x + 9 = 0$, by writing -9, $+15$, -27, and $+9$ for p, q, r, and s respectively, there will come out $-5\frac{1}{4} = \alpha$, $-50\frac{5}{8} = \beta$, and $2\frac{7}{64} = \zeta$. The common Divisors of β and 2ζ, or $-\frac{405}{8}$ and $\frac{135}{32}$ are 3, 5, 9, 15, 27, 45, and 135; but 9 is a Square Number, and 3, 15, 27, 135, divided by the Number 4, do not leave Unity, as, by reason of the odd Term p they ought to do. These therefore being rejected, there remain only 5 and 45 to be try'd for n. Let us put therefore, first $n = 5$, and the odd Divisors of $\frac{\beta}{n}$ or $-\frac{81}{8}$ being halv'd, *viz.* $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, $\frac{27}{2}$, $\frac{81}{2}$, are to be try'd for k. If k be made $\frac{1}{2}$, the Quotient $-\frac{81}{4}$, which comes out by dividing $\frac{\beta}{n}$ by k, subtracted from $\frac{1}{2}pk$, or $-\frac{9}{4}$, leaves 18 for l, and $\frac{\alpha + nkk}{2}$, or -2, is Q, and $QQ - s$, or -5, may be divided by n, or 5; but the Root of the Negative Quotient -1 is impossible, which yet ought to be 18. Wherefore I conclude k not to be $\frac{1}{2}$, and then I try if it be $\frac{3}{2}$. The Quotient which arises by dividing $\frac{\beta}{n}$ by k, or $-\frac{81}{8}$ by $\frac{3}{2}$, *viz.* the Quotient $-\frac{27}{4}$ I subtract from $\frac{1}{2}pk$, or $-\frac{27}{4}$, and there remains 0; whence now l will be nothing. But $\frac{\alpha + nkk}{2}$, or 3, is equal to Q, and $QQ - s$ is nothing; whence again l, which is the Root of $QQ - s$, divided by n, is found to be nothing. Wherefore these Things thus agreeing, I conclude n to be $= 5$, $k = \frac{3}{2}$, $l = 0$, and $Q = 3$; and therefore by adding to each Part of the Æquation propos'd, the Terms $nkkxx + 2nlkx + nll$, that is, $\frac{45}{4}xx$, and by extracting on both Sides the Square Root, there comes out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, that is, $xx - 4\frac{1}{2}x + 3 = \sqrt{5} \times \frac{3}{2}x$.

By the same Method, Literal Æquations are also reduc'd. As if there was $x^4 - 2ax^3 {+2aa \atop -cc} xx - 2a^3x + a^4 = 0$.

by

by substituting $-2a$, $2aa-cc$, $-2a^3$, and $+a^4$ for p, q, r, and s respectively, you obtain $aa-cc=\alpha$, $-acc-a^3=\beta$, and $\frac{3}{4}a^4+\frac{1}{2}aacc-\frac{1}{4}c^4=\zeta$. The common Divisor of the Quantities β and 2ζ is $aa+cc$, which then will be n; and $\frac{\beta}{n}$ or $-a$, has the Divisors 1 and a. But because n is of two Dimensions, and $k\sqrt{n}$ ought to be of no more than one, therefore k will be of none, and consequently cannot be a. Let therefore k be 1, and $\frac{\beta}{n}$ being divided by k, take the Quotient $-a$ from $\frac{1}{2}pk$, and there will remain nothing for l. Moreover, $\frac{\alpha+nkk}{2}$, or aa, is Q, and $QQ-s$, or a^4-a^4, is 0; and thence again there comes out nothing for l. Which shews the Quantities n, k, l, and Q, to be rightly found; and adding to each Part of the Æquation propos'd, the Terms $nkkxx+2nklx+nll$, that is, $aaxx+ccxx$, the Root may be extracted on both Sides; and by that Extraction there will come out $xx+\frac{1}{2}px+Q=\sqrt{n}\times\overline{kx+l}$, that is, $xx-ax+aa=\pm x\sqrt{aa+cc}$. And the Root being again extracted, you'll have $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{aa+cc}\pm\sqrt{\frac{1}{4}cc-\frac{1}{2}aa\pm\frac{1}{2}a\sqrt{aa+cc}}$.

Hitherto I have apply'd the Rule to the Extraction of Surd Roots; the same may also be apply'd to the Extraction of Rational Roots, if for the Quantity n you make Use of Unity: and after that Manner we may examine, whether an Æquation that wants Fracted or Surd Terms can admit of any Divisor, either Rational or Surd, of two Dimensions. As if the Æquation $x^4-x^3-5xx+12x-6=0$ was propos'd, by substituting -1, -5, $+12$, and -6 for p, q, r, and s respectively, you'll find $-5\frac{1}{4}=\alpha$, $9\frac{3}{8}=\beta$, and $-10\frac{57}{64}=\zeta$. The common Divisor of the Terms β and 2ζ, or of $\frac{75}{8}$ and $-\frac{825}{32}$ is only Unity. Wherefore I put $n=1$. The Divisors of the Quantity $\frac{\beta}{n}$, or $\frac{75}{8}$, are 1, 3, 5, 15, 25, 75; the Halves whereof (if p be odd) are to be try'd for k. And if for k we try $\frac{5}{2}$, you'll have $\frac{1}{2}pk-\frac{\beta}{nk}=-5$, and its half $-\frac{5}{2}=l$. Also

$\alpha+nkk$

$\frac{\alpha + nkk}{2} = \frac{1}{2} = Q$, and $\frac{QQ - s}{n} = 6\frac{1}{4}$, the Root whereof agrees with l. I therefore conclude, that the Quantities n, k, l, Q are rightly found; and having added to each Part of the Æquation the Terms $nkkxx + 2nklx + nll$, that is, $6\frac{1}{4}xx - 12\frac{1}{2}x + 6\frac{1}{4}$, the Root may be extracted on both Sides; and by that Extraction there will come out $xx + \frac{1}{2}px + Q = \pm\sqrt{n} \times \overline{kx + l}$, that is, $xx - \frac{1}{2}x + \frac{1}{2} = \pm 1 \times 2\frac{1}{2}x - 2\frac{1}{2}$, or $xx - 3x + 3 = 0$, and $xx + 2x - 2 = 0$, and so by these two Quadratick Æquations the Biquadratick one propos'd may be divided. But Rational Divisors of this Sort may more expeditiously be found by the other Method deliver'd above.

If at any Time there are many Divisors of the Quantity $\frac{\beta}{n}$, so that it may be too difficult to try all of them for k, their Number may be soon diminish'd, by seeking all the Divisors of the Quantity, $\alpha s - \frac{1}{4}rr$. For the Quantity Q ought to be equal to some one of these, or to the half of some odd one. Thus, in the last Example, $\alpha s - \frac{1}{4}rr$ is $-\frac{9}{2}$, some one of whose Divisors, 1, 3, 9, or of them halv'd $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}$, ought to be Q. Wherefore, by trying singly the halv'd Divisors of the Quantity $\frac{\beta}{n}$, *viz.* $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2}, \frac{25}{2}$, and $\frac{75}{2}$ for k, I reject all that do not make $\frac{1}{2}\alpha + \frac{1}{2}nkk$, or $-\frac{21}{8} + \frac{1}{2}kk$; that is, Q is one of the Numbers 1, 3, 9, $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}$. But by writing $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2}$, &c. for k, there come out respectively $-\frac{5}{2}, -\frac{3}{2}, +\frac{1}{2}, +\frac{51}{2}$, &c. for Q; out of which only $-\frac{3}{2}$ and $\frac{1}{2}$ are found among the aforesaid Numbers 1, 3, 9, $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}$, and consequently the rest being rejected, either k will be $= \frac{3}{2}$ and $Q = -\frac{3}{2}$, or $k = \frac{5}{2}$ and $Q = \frac{1}{2}$. Which two Cases are examin'd. And so much of Æquations of four Dimensions.

If an Æquation of six Dimensions is to be reduc'd, let it be $x^6 + px^5 + qx^4 + rx^3 + sxx + tx + v = 0$, and make

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}p\alpha = \beta. \quad s - \tfrac{1}{2}p\beta = \gamma.$$
$$\gamma - \tfrac{1}{4}\alpha\alpha = \zeta. \quad t - \tfrac{1}{2}\alpha\beta = \eta. \quad v - \tfrac{1}{4}\beta\beta = \theta.$$
$$\zeta\theta - \tfrac{1}{4}\eta\eta = \lambda.$$

Then take for n some common Integer Divisor, that is not a Square, out of the Terms $2\zeta, \eta, 2\theta$, and that likewise is not

not divisible by a Square Number, and which also divided by the Number 4, shall leave Unity; if but any one of the Terms p, r, t be odd. For k take some Integer Divisor of the Quantity $\frac{\lambda}{2nn}$ if p be even; or the half of an odd Divisor if p be odd; or 0 if λ be 0. For Q [take] the Quantity $\frac{1}{2}\alpha + \frac{1}{2}nkk$. For l some Divisor of the Quantity $\frac{Qr - QQp - t}{n}$ if Q be an Integer; or the half of an odd Divisor, if Q be a Fraction that has for its Denominator the Number 2; or 0, if the Dividual [or the Quantity] $\frac{Qr - QQp - t}{n}$ be nothing. And for R the Quantity $\frac{1}{2}r - \frac{1}{2}Qp + \frac{1}{2}nkl$. Then try if $RR - v$ can be divided by n, and the Root of the Quotient extracted; and besides, if that Root be equal as well to the Quantity $\frac{QR - \frac{1}{2}t}{nl}$ as to the Quantity $\frac{QQ + pR - nll - s}{2nk}$. If all these happen, call that Root m; and in room of the Æquation propos'd, write this, $x^3 + \frac{1}{2}pxx + Qx + r = \pm\sqrt{n} \times \overline{kxx + lx + m}$. For this Æquation, by squaring its Parts, and taking from both Sides the Terms on the Right-Hand, will produce the Æquation propos'd. Now if all these Things do not happen in the Case propos'd, the Reduction will be impossible, if it appears beforehand that the Æquation cannot be reduc'd by a rational Divisor.

For Example, let there be propos'd the Æquation $x^6 -$

$$2ax^5 + 2bbx^4 + 2abbx^3 \begin{matrix} -2aabb \\ +2a^3b \\ -4ab^3 \end{matrix}\; xx \begin{matrix} +3aab^4 \\ -a^4bb \end{matrix} = 0,$$

and by writing $-2a$, $+2bb$, $+2abb$, $-2aabb + 2a^3b - 4ab^3$, 0, and $3aab^4 - a^4bb$ for p, q, r, s, t, and v respectively, there will come out $2bb - aa = \alpha$. $4abb - a^3 = \beta$. $2a^3b + 2aabb - 4ab^3 - a^4 = \gamma$. $-b^4 + 2a^3b + 3aabb - 4ab^3 - \frac{5}{4}a^4 = \zeta$. $\frac{1}{2}a^5 - a^3bb = \eta$, and $3aab^4 - a^4bb - \frac{1}{4}a^6 = \theta$. And the common Divisor of the Terms 2ζ, η, and 2θ, is $aa - 2bb$, or $2bb - aa$, according as aa or $2bb$ is the greater. But let aa be greater than $2bb$, and $aa - 2bb$ will be n. For n must always be Affirmative. Moreover, $\frac{\zeta}{n}$ is $-\frac{5}{4}aa + 2ab + \frac{1}{2}db$, $\frac{\eta}{n}$ is

$\frac{1}{2}a^3$,

$\frac{1}{2}a^3$, and $\frac{\theta}{n}$ is $-\frac{1}{4}a^4-\frac{1}{2}aabb$, and conſequently $\frac{\zeta}{2n}\times\frac{\theta}{n}-\frac{\eta\eta}{4nn}$, or $\frac{\lambda}{2nn}$, is $\frac{1}{8}a^6-\frac{1}{4}a^5b+\frac{1}{4}a^4bb-\frac{1}{2}a^3b^3-\frac{1}{4}aab^4$, the Diviſors whereof are 1, a, aa; but becauſe $\sqrt{n}\times k$ cannot be of more than one Dimenſion, and the $\sqrt{n}$ is of one, therefore k will be of none; and conſequently can only be a Number. Wherefore, rejecting a and aa, there remains only 1 for k. Beſides, $\frac{1}{2}\alpha+\frac{1}{2}nkk$ gives 0 for Q, and $\frac{Qr-QQp-t}{n}$ is alſo nothing; and conſequently l, which ought to be its Diviſor, will be nothing. Laſtly, $\frac{1}{2}r-\frac{1}{2}pQ+\frac{1}{2}nkl$ gives abb for R. And $RR-v$ is $-2aab^4+a^4bb$, which may be divided by n, or $aa-2bb$, and the Root of the Quotient $aabb$ be extracted, and that Root taken Negatively, *viz.* $-ab$, is not unequal to the indefinite Quantity $\frac{QR-\frac{1}{2}t}{nl}$, or $\frac{0}{0}$, but equal to the definite Quantity $\frac{QQ+pR-nll-s}{2nk}$. Wherefore that Root $-ab$ will be m, and in the room of the Æquation propos'd, there may be writ $x^3-\frac{1}{2}pxx+Qx+R=\sqrt{n}\times\overline{kxx+lx+m}$, that is, $x^3-axx+abb=\sqrt{aa-2bb}\times\overline{xx-ab}$. The Truth of which Æquation you may prove by ſquaring the Parts of the Æquation found, and taking away the Terms on the Right Hand from both Sides. From that Operation will be produc'd the Æquation $x^6-2ax^5+2bbx^4+2abbx^3-2aabbxx+2a^3bxx-4ab^3xx+3aab^4-a^4bb=0$, which was to be reduc'd.

If the Æquation is of eight Dimenſions, let it be $x^8+px^7+qx^6+rx^5+sx^4+tx^3+vxx+wx+z=0$, and make $q-\frac{1}{4}pp=\alpha$. $r-\frac{1}{2}p\alpha=\beta$. $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$. $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$. $v-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$. $w-\frac{1}{2}\beta\gamma=\zeta$, and $z-\frac{1}{4}\gamma\gamma=\eta$. And ſeek a common Diviſor of the Terms 2δ, 2ε, 2ζ, 8η, that ſhall be an Integer, and neither a Square Number, nor diviſible by a Square Number; and which alſo divided by 4 ſhall leave Unity, if any of the alternate Terms p, r, t, w be odd. If there be no ſuch common Diviſor, it is certain that the Æquation cannot be reduc'd by the Extraction of a Quadratick Surd Root,

 and

and if it cannot be so reduc'd, there will scarce be found a common Divisor of all those four Quantities. The Operation therefore hitherto is a Sort of an Examination, whether the Æquation be reducible or not; and consequently, since that Sort of Reductions are seldom possible, it will most commonly end the Work.

And, by a like Reason, if the Æquation be of ten, twelve, or more Dimensions, the Impossibility of its Reduction may be known. As if it be $x^{10} + px^9 + qx^8 + rx^7 + sx^6 + tx^5 + vx^4 + ax^3 + bx^2 + cx + d = 0$, you must make $q - \frac{1}{4}pp = \alpha$, $r - \frac{1}{2}p\alpha = \beta$, $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$, $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = \delta$, $v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \varepsilon$, $a - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma = \zeta$, $b - \frac{1}{2}\beta\delta - \frac{1}{4}\gamma\gamma = \eta$, $c - \frac{1}{2}\gamma\delta = \theta$, $d - \frac{1}{4}\delta\delta = \kappa$. And seek such a common Divisor to the five Terms, 2ε, 2ζ, 8η, 4θ, 8κ, as is an Integer, and not a Square, but which shall leave 1 when divided by 4, if any one of the Terms p, r, t, a, c be odd.

So if there be an Æquation of twelve Dimensions, as $x^{12} + px^{11} + qx^{10} + rx^9 + sx^8 + tx^7 + vx^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, make $q - \frac{1}{4}pp = \alpha$, $r - \frac{1}{2}p\alpha = \beta$, $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$, $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = \delta$, $v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \varepsilon\varepsilon$, $a - \frac{1}{2}p\varepsilon - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma = \zeta$, $b - \frac{1}{2}\alpha\varepsilon - \frac{1}{2}\beta\delta - \frac{1}{4}\gamma\gamma = \eta$, $c - \frac{1}{2}\beta\varepsilon - \frac{1}{2}\gamma\delta = \theta$, $d - \frac{1}{2}\gamma\varepsilon - \frac{1}{4}\delta\delta = \kappa$, $e - \frac{1}{2}\delta\varepsilon = \lambda$, $f - \frac{1}{4}\varepsilon\varepsilon = \mu$, and you must seek a common Integer Divisor of the six Terms 2ζ, 8η, 4θ, 8κ, 4λ, 8μ, that is not a Square, but being divided by 4 shall leave Unity, if any one of the Terms p, r, t, a, c, e be odd.

And thus you may go on *ad infinitum*, and the propos'd Æquation will be always irreduceable when it has no common Divisor. But if at any Time such a Divisor n being found, there are Hopes of a future Reduction, and it may be found by working or following the Steps of the Operation we shew'd in the Æquation of eight Dimensions.

Seek a Square Number, to which after it is multiply'd by n, the last Term z of the Æquation being added under its proper Sign, shall make a Square Number. But that may be expeditiously perform'd if you add to z, where n is an even Number, or to $4z$ when it is odd, these Quantities successively n, $3n$, $5n$, $7n$, $9n$, $11n$, and so on till the Sum becomes equal to some Number in the Table of Square

Num-

Numbers, which I suppose to be ready at Hand. And if no such Square Number occurs before the Square Root of that Sum, augmented by the Square Root of the Excess of that Sum above the last Term of the Æquation, is four times greater than the greatest of the Terms of the propos'd Æquation $p, q, r, s, t, v,$ &c. there will be no Occasion to try any farther. For then the Æquation cannot be reduc'd. But if such a Square Number does accordingly occur, let its Root be S if n is even, or $2S$ if n be odd; and call the $\sqrt{\frac{SS - z}{n}} = h$. But s and h ought to be Integers if n is even, but if n is odd, they may be Fractions that have 2 for their Denominator. And if one is a Fraction, the other ought to be so too. Which also is to be observ'd of the Numbers R and M, Q and l, P and k hereafter to be found. And all the Numbers S and h, that can be found within the prescrib'd Limit, must be collected in a [Table or] Catalogue.

Afterwards, for (k) all the Numbers are to be successively try'd, which do not make $nk \pm \frac{1}{2}p$ four times greater than the greatest Term of the Æquation, and you must in all Cases put $\frac{nkk + \alpha}{2} = Q$. Then you are to try successively for l all the Numbers that do not make $nl \pm Q$ four times greater than the greatest Term of the Æquation; and in every Tryal put $\frac{-npkk + 2\beta}{4} + nkl = R$. Lastly, for m you must try successively all the Numbers which do not make $nm + R$ four times greater than the greatest of the Terms of the Æquation, and you must see whether in any Case if you make $s - QQ - PR + nll = 2H$, and $H + nkm = S$, let S be some of the Numbers which were before brought into the Catalogue for S; and besides, if the other Number answering to that S, which being set down for h in the same Catalogue, will be equal to these three, $\frac{2RS - w}{2nm}$, $\frac{2QS + RR - v - nmm}{2nl}$, and $\frac{PS + 2QR - t - 2nlm}{2nk}$. If all these Things shall happen in any Case, instead of the Æquation propos'd, you must write this $x^4 + \frac{1}{2}px^3 + Qxx + Rx + S = \sqrt{n} \times \overline{kx^3 + lxx + mx + h}$.

 For

For Example, let there be propos'd the Æquation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x - 5 = 0$, and you'll have $q - \frac{1}{4}pp = -1 - 4 = -5 = \alpha$. $r - \frac{1}{2}p\alpha = -10 + 10 = 0 = \beta$. $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = 5 - \frac{25}{4} = -\frac{5}{4} = \gamma$. $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = -5 + \frac{5}{2} = -\frac{5}{2} = \delta$. $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = -10 - \frac{25}{8} = -\frac{105}{8}$. $w - \frac{1}{2}\beta\gamma = -10 = \zeta$. $z - \frac{1}{4}\gamma\gamma = -5 - \frac{25}{64} = -\frac{345}{64} = \eta$. Therefore $2\delta, 2\varepsilon, 2\zeta, 8\eta$ respectively are -5; $-\frac{105}{4}$, -20, and $-\frac{345}{8}$, and their common Divisor 5, which divided by 4, leaves 1, as it ought, because the Term s is odd. Since therefore the common Divisor n, or 5, is found, which gives hope to a future Reduction, and because it is odd to $4z$, or -20, I successively add n, $3n$, $5n$, $7n$, $9n$, &c. or 5, 15, 25, 35, 45, &c. and there arises -15, 0, 25, 60, 105, 160, 225, 300, 385, 480, 585, 700, 825, 960, 1105, 1260, 1425, 1600. Of which only 0, 25, 225, and 1600 are Squares. And the Halves of these Roots 0, $\frac{5}{2}$, $\frac{15}{2}$, 20, collect in a Table for the Values of S, and so the Values of $\sqrt{\frac{SS - z}{n}}$, that is, 1, $\frac{3}{2}$, $\frac{7}{2}$, 9, for h. But because $S + nh$, if 20 be taken for S and 9 for h, becomes 65, a Number greater than four times the greatest Term of the Æquation, therefore I reject 20 and 9, and write only the rest in the Table as follows:

h	1	$\frac{3}{2}$	$\frac{7}{2}$
S	0	$\frac{5}{2}$	$\frac{15}{2}$

Then try for k all the Numbers which do not make $\frac{1}{2} \pm nk$, or $2 \pm 5k$, greater than 40, (four times the greatest Term of the Æquation) that is, the Numbers -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, putting $\frac{nkk + \alpha}{2}$, or $\frac{5kk - 5}{2}$, that is, the Numbers $\frac{315}{2}$, 120, $\frac{175}{2}$, 60, $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, 60, $\frac{175}{2}$, 120, respectively for Q. But even when $Q \pm nl$, and much more Q, ought not to be greater than 40, I perceive I am to reject $\frac{315}{2}$, 120, $\frac{175}{2}$, and 60, and their Correspondents -8, -7, -6, -5, 5, 6, 7, and consequently that only -4, -3, -2, -1, 0, 1, 2, 3, 4, must respectively be try'd for k, and $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, respectively for Q. Let us therefore try -1 for k, and 0 for Q, and in this Case for l there will be successively to be try'd all the Numbers which do not make $Q \pm nl$ greater than

40,

40, that is, all the Numbers between 10 and -10; and for R you are respectively to try the Numbers $\frac{2\beta - npkk}{4}$ $+ nkl$, or $-5, -5l$, that is, $-55, -50, -45, -40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 35, 40, 45$, the three former of which and the last, because they are greater than 40, may be neglected. Let us try therefore -2 for l, and 5 for R, and in this Case for m there will be besides to be try'd all the Numbers which do not make $R \pm mn$, or $5 \pm mn$, greater than 40, that is, all the Numbers between 7 and -9, and see whether or not by putting $s - QQ - pR + nll$, that is $5 - 20 + 20$, or $5 = 2H$, let $H + nkm$, or $\frac{5}{2} - 5m = S$, that is, if any of these Numbers $\frac{-65}{2}, \frac{-55}{2}, \frac{-45}{2}, \frac{-35}{2}, \frac{-25}{2}, \frac{-15}{2}, \frac{-5}{2}, \frac{5}{2}, \frac{15}{2}, \frac{25}{2}, \frac{35}{2}, \frac{45}{2}, \frac{55}{2}, \frac{65}{2}, \frac{75}{2}, \frac{85}{2}$, is equal to any of the Numbers $0, \pm\frac{5}{2}, \pm\frac{15}{2}$, which were first brought into the Catalogue for S. And we meet with four of these $-\frac{15}{2}, -\frac{5}{2}, \frac{5}{2}, \frac{15}{2}$, to which answer $\pm\frac{7}{2}, \pm\frac{3}{2}, \pm\frac{3}{2}, \pm\frac{7}{2}$, being writ for h in the same Table, as also $2, 1, 0, -1$ substituted for m. But let us try $-\frac{5}{2}$ for S, 1 for m, and $\pm\frac{3}{2}$ for h, and you'll have $\frac{2RS - w}{2nm} = \frac{-25 + 10}{10} = -\frac{3}{2}$, and $\frac{2QS + RR - vnmm}{2nl} = \frac{25 + 10 - 5}{-20} = -\frac{3}{2}$, and $\frac{pS + 2QR - t - 2nlm}{2nk} = \frac{-10 + 5 + 20}{-10} = -\frac{3}{2}$. Wherefore, since there comes out in all Cases $-\frac{3}{2}$, or h, I conclude all the Numbers to be rightly found, and consequently that in room of the Æquation propos'd, you must write $x^4 + \frac{1}{2}px^3 + Qxx + Rx + S = \sqrt{n} \times \overline{kx^3 + lxx + mx + h}$, that is, $x^4 + 2x^3 + 5x - 2\frac{1}{2} = \sqrt{5} \times \overline{-x^3 - 2xx + x - 1\frac{1}{2}}$. For by squaring the Parts of this, there will be produc'd that Æquation of eight Dimensions, which was at first propos'd.

Now, if by trying all the Cases of the Numbers, all the aforesaid Values of h do not in any Case consent, it would be an Argument that the Æquation could not be solv'd by the Extraction of the Surd Quadratick Root.

I might

I might now join the Reductions of Æquations by the Extraction of the Surd Cubick Root, but these, as being seldom of Use, I pass by. Yet there are some Reductions of Cubick Æquations commonly known, which, if I should wholly pass over, the Reader might perhaps think us deficient. Let there be propos'd the Cubick Æquation $x^3 * + qx + r = 0$; the second Term whereof is wanting: For that every Cubick Æquation may be reduc'd to this Form, is evident from what we have said above. Let x be suppos'd $= a + b$. Then will $a^3 + 3aab + 3abb + b^3$ (that is x^3) $+ qx + r = 0$. Let $3aab + 3abb$ (that is, $3abx$) $+ qx = 0$, and then will $a^3 + b^3 + r = 0$. By the former Æquation b is $= -\frac{q}{3a}$, and cubically $b^3 = -\frac{q^3}{27a^3}$. Therefore by the latter, $a^3 - \frac{q^3}{27a^3} + r = 0$, or $a^6 + ra^3 = \frac{q^3}{27}$, and by the Extraction of the affected Quadratick Root, $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Extract the Cubick Root and you'll have a. And above, you had $\frac{-q}{3a} = b$, and $a + b = x$. Therefore $a - \frac{q}{3a}$ is the Root of the Æquation propos'd.

For Example, let there be propos'd the Æquation $y^3 - 6yy + 6y + 12 = 0$. To take away the second Term of this Æquation, make $x + 2 = y$, and there will arise $x^3 * - 6x + 8 = 0$. Where q is $= -6$, $r = 8$, $\frac{1}{4}rr = 16$, $\frac{q^3}{27} = -8$, $a^3 = -4 \pm \sqrt{8}$, $a - \frac{q}{3a} = x$, and $x + 2 = y$, that is, $2 + \sqrt[3]{-4 \pm \sqrt{8}} + \frac{2}{\sqrt[3]{-4 \pm \sqrt{8}}}$.

And after this Way the Roots of all Cubical Æquations may be extracted wherein q is Affirmative; or also wherein q is Negative, and $\frac{q^3}{27}$ not greater than $\frac{1}{4}rr$, that is, wherein two of the Roots of the Æquation are impossible. But where q is Negative, and $\frac{q^3}{27}$ at the same time greater

than

than $\frac{1}{4}rr$, $\sqrt{\frac{1}{4}rr - \frac{q^3}{27}}$ becomes an impoſſible Quantity; and ſo the Root of the Æquation x or y will, in this Caſe, be impoſſible, *viz.* in this Caſe there are three poſſible Roots, which all of them are alike with reſpect to the Terms of the Æquations q and r, and are indifferently denoted by the Letters x and y, and conſequently all of them may be extracted by the ſame Method, and expreſs'd the ſame Way as any one is extracted or expreſs'd; but it is impoſſible to expreſs all three by the Law aforeſaid. The Quantity $a - \frac{q}{3a}$, whereby x is denoted, cannot be manyfold, and for that Reaſon the Suppoſition that x, in this Caſe wherein it is triple, may be equal to the Binomial $a - \frac{q}{3a}$, or $a + b$, the Cubes of whoſe Terms $a^3 + b^3$ are together $= r$, and the triple Rectangle $3ab$ is $= q$, is plainly impoſſible; and it is no Wonder that from an impoſſible Hypotheſis, an impoſſible Concluſion ſhould follow.

There is, moreover, another Way of expreſſing theſe Roots, *viz.* from $a^3 + b^3 + r$, that is, from nothing take $a^3 + r$, or $\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$, and there will remain $b^3 = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. Therefore a is $=$

$\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and $b =$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$; or $a =$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and $b =$

$\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$, and conſequently the Sum of

theſe $\sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}} +$

$\sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}}$ will be x.

Moreover, the Roots of Biquadratick Æquations may be extracted and expreſs'd by means of Cubick ones. But firſt you muſt take away the ſecond Term of the Æquation. Let the Æquation that [then] reſults be $x^4 + qxx + rx + s = 0$. Suppoſe this to be generated by the Multiplication of theſe two $xx + ex + f = 0$, and $xx - ex + g = 0$, that is, to be the ſame with this $x^4 * \begin{matrix} +f \\ +g \\ -ee \end{matrix} xx \begin{matrix} +eg \\ -ef \end{matrix} x + fg = 0$, and comparing the Terms you'll have $f + g - ee = q$, $eg - ef = r$, and $fg = s$. Wherefore $q + ee = f + g$, $\frac{r}{e} = g - f$, $\frac{q + ee + \frac{r}{e}}{2} = g$, $\frac{q + ee - \frac{r}{e}}{2} = f$, $\frac{qq + 2eeq + e^4 - \frac{rr}{ee}}{4} (= fg) = s$, and by Reduction $e^6 + 2qe^4 \begin{matrix} +qq \\ -4s \end{matrix} ee - rr = 0$. For ee write y, and you'll have $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$, a Cubick Æquation, whoſe ſecond Term may be taken away, and then the Root extracted either by the precedent Rule or otherwiſe. Then that Root being had, you muſt go back again, by putting $\sqrt{y} = e$, $\frac{q + ee - \frac{r}{e}}{2} = f$, $\frac{q + ee + \frac{r}{e}}{2} = g$, and the two Æquations $xx + ex + f = 0$, and $xx - ex + g = 0$, their Roots being extracted, will give the four Roots of the Biquadratick Æquation $x^4 + qxx + rx + s = 0$, *viz.* $x = -\frac{1}{2}e \pm \sqrt{\frac{1}{4}ee - f}$, and $x = \frac{1}{2}e \pm \sqrt{\frac{1}{4}ee - g}$. Where note, that if the four Roots of the Biquadratick Æquation are poſſible, the three Roots of the Cubick Æquation $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$ will be poſſible alſo, and conſequently cannot be extracted by the precedent Rule. And thus, if the affected Roots of an Æquation of five or more Dimenſions are converted into Roots that are not affected, the middle Terms of the Æquation being taken away, that Expreſſion of the Roots will be always impoſſible, where more

more than one Root in an Æquation of odd Dimensions are possible, or more than two in an Æquation of even Dimensions, which cannot be reduc'd by the Extraction of the Surd Quadratick Root, by the Method laid down above.

Monsieur *Des Cartes* taught how to reduce a Biquadratick Æquation by the Rules last deliver'd. *E.g.* Let there be propos'd the Æquation reduc'd above, $x^4 - x^3 - 5xx + 12x - 6 = 0$. Take away the second Term, by writing $v + \frac{1}{4}$ for x, and there will arise $v^4 - \frac{43}{8}vv + \frac{75}{8}v - \frac{851}{256} = 0$. To take away the Fractions, write $\frac{1}{4}z$ for v, and there will arise $z^4 - 86zz + 600z - 851 = 0$. Here $-86 = q$, $600 = r$, and $-851 = s$, and consequently $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$, and substituting what is equivalent, you'll have $y^3 - 172yy + 10800y - 360000 = 0$. Where trying all the Divisors of the last Term $1, -1, 2, -2, 3, -3, 4, -4, 5, -5$, and so onwards to 100, you'll find at length $y = 100$. Which yet may be found far more expeditiously by our Method above deliver'd. Then having got y, its Root 10 will be e, and $\dfrac{q + ee - \frac{r}{e}}{2}$, that is, $\dfrac{-86 + 100 - 60}{2}$, or -23, will be f, and $\dfrac{q + ee + \frac{r}{e}}{2}$, or 37 will be g, and consequently the Æquations $xx + ex + f = 0$, and $xx - ex + g = 0$, and writing z for x, and substituting equivalent Quantities, will become $zz + 10z - 23 = 0$, and $zz - 10z + 37 = 0$. Restore v in the room of $\frac{1}{4}z$, and there will arise $vv + 2\frac{1}{2}v - \frac{23}{16} = 0$, and $vv - 2\frac{1}{2}v + \frac{37}{16} = 0$. Restore, moreover, $x - \frac{1}{4}$ for v, and there will come out $xx + 2x - 2 = 0$, and $xx - 3x + 3 = 0$, two Æquations; the four Roots whereof $x = -1 \pm \sqrt{3}$, and $x = 1\frac{1}{2} + \sqrt{-\frac{3}{4}}$, are the same with the four Roots of the Biquadratick Æquation propos'd at the Beginning, $x^4 - x^3 - 5xx + 12x - 6 = 0$. But these might have been more easily found by the Method of finding Divisors, explain'd before.

Hitherto it will suffice, I suppose, to have given the Reductions of Æquations after a more easy and more general Way than what has been done by others. But since among these Operations we often meet with complex radical Quantities, which may be reduc'd to more simple ones, it is convenient to explain the Reduction of those also. They are perform'd by the Extractions of Roots out of Binomial Quantities, or out of Quantities more compounded, which may be consider'd as Binomial ones.

The Extraction of Roots out of Binomial Quantities.

[But since this is already done in the Chapter of the *Reduction of Radicals to more simple Radicals, by means of the Extraction of Roots*, we shall say no more of it here.]

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Linear Construction

OF

ÆQUATIONS.

HITHERTO I have shewn the Properties, Transmutations, Limits, and Reductions of all Sorts of Æquations. I have not always joyn'd the Demonstrations, because they seem'd too easy to need it, and sometimes cannot be laid down without too much Tediousness. It remains now only to shew, how, after Æquations are reduc'd to their most commodious Form, their Roots may be extracted in Numbers. And here the chief Difficulty lies in obtaining the two or three first Figures; which may be most commodiously done by either the Geometrical or Mechanical Construction of an Æquation. Wherefore I shall subjoin some of these Constructions.

The Antients, as we learn from *Pappus*, in vain endeavour'd at the Trisection of an Angle, and the finding out of two *mean Proportionals* by a right Line and a Circle. Afterwards they began to consider the Properties of several other Lines, as the Conchoid, the Cissoid, and the Conick Sections, and by some of these to solve those Problems. At length, having more throughly examin'd the Matter, and the Conick Sections being receiv'd into Geometry, they distinguish'd Problems into three Kinds, *viz.* (1.) Into Plane ones, which deriving their Original from Lines on a Plane, may be solv'd by a right Line and a Circle; (2.) Into Solid ones, which were solved by Lines deriving their Original

nal from the Consideration of a Solid, that is, of a Cone; (3.) And Linear ones, to the Solution of which were requir'd Lines more compounded. And according to this Distinction we are not to solve solid Problems by other Lines than the Conick Sections; especially if no other Lines but right ones, a Circle, and the Conick Sections, must be receiv'd into Geometry. But the Moderns advancing yet much farther, have receiv'd into Geometry all Lines that can be express'd by Æquations, and have distinguish'd, according to the Dimensions of the Æquations, those Lines into Kinds; and have made it a Law, that you are not to construct a Problem by a Line of a superior Kind, that may be constructed by one of an inferior one. In the Contemplation of Lines, and finding out their Properties, I like their Distinction of them into Kinds, according to the Dimensions of the Æquations by which they are defin'd. But it is not the Æquation, but the Description that makes the Curve to be a Geometrical one. The Circle is a Geometrical Line, not because it may be express'd by an Æquation, but because its Description is a Postulate. It is not the Simplicity of the Æquation, but the Easiness of the Description, which is to determine the Choice of our Lines for the Construction of Problems. For the Æquation that expresses a Parabola, is more simple than That that expresses a Circle, and yet the Circle, by reason of its more simple Construction, is admitted before it. The Circle and the Conick Sections, if you regard the Dimension of the Æquations, are of the same Order, and yet the Circle is not number'd with them in the Construction of Problems, but by reason of its simple Description, is depress'd to a lower Order, *viz.* that of a right Line; so that it is not improper to express that by a Circle that may be express'd by a right Line. But it is a Fault to construct that by the Conick Sections which may be constructed by a Circle. Either therefore you must take your Law and Rule from the Dimensions of Æquations as observ'd in a Circle, and so take away the Distinction between Plane and Solid Problems; or else you must grant, that that Law is not so strictly to be observ'd in Lines of superior Kinds, but that some, by reason of their more simple Description, may be preferr'd to others of the same Order, and may be number'd with Lines of inferior Orders in the Construction of Problems. In Constructions that are equally Geometrical, the most simple are always to be preferr'd. This Law is so universal, as to be without Exception,

ception. But Algebraick Expressions add nothing to the Simplicity of the Construction; the bare Descriptions of the Lines only are here to be consider'd; and these alone were consider'd by those Geometricians who joyn'd a Circle with a right Line. And as these are easy or hard, the Construction becomes easy or hard: And therefore it is foreign to the Nature of the Thing, from any Thing else to establish Laws about Constructions. Either therefore let us, with the Antients, exclude all Lines besides the Circle, and perhaps the Conick Sections, out of Geometry, or admit all, according to the Simplicity of the Description. If the Trochoid were admitted into Geometry, we might, by its Means, divide an Angle in any given Ratio. Would you therefore blame those who should make Use of this Line to divide an Angle in the Ratio of one Number to another, and contend that this Line was not defin'd by an Æquation, but that you must make Use of such Lines as are defin'd by Æquations? If therefore, when an Angle was to be divided, for Instance, into 10001 Parts, we should be oblig'd to bring a Curve defin'd by an Æquation of above an hundred Dimensions to do the Business; which no Mortal could describe, much less understand; and should prefer this to the Trochoid, which is a Line well known, and describ'd easily by the Motion of a Wheel or a Circle, who would not see the Absurdity? Either therefore the Trochoid is not to be admitted at all into Geometry, or else, in the Construction of Problems, it is to be preferr'd to all Lines of a more difficult Description. And there is the same Reason for other Curves. For which Reason we approve of the Trisections of an Angle by a Conchoid, which *Archimedes* in his Lemma's, and *Pappus* in his Collections, have preferr'd to the Inventions of all others in this Case; because we ought either to exclude all Lines, besides the Circle and right Line, out of Geometry, or admit them according to the Simplicity of their Descriptions, in which Case the Conchoid yields to none, except the Circle. Æquations are Expressions of Arithmetical Computation, and properly have no Place in Geometry, except as far as Quantities truly Geometrical (that is, Lines, Surfaces, Solids, and Proportions) may be said to be some equal to others. Multiplications, Divisions, and such sort of Computations, are newly receiv'd into Geometry, and that unwarily, and contrary to the first Design of this Science. For whosoever considers the Construction of Problems by a right Line and a Circle, found out by the first

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Geometricians, will easily perceive that Geometry was invented that we might expeditiously avoid, by drawing Lines, the Tediousness of Computation. Therefore these two Sciences ought not to be confounded. The Antients did so industriously distinguish them from one another, that they never introduc'd Arithmetical Terms into Geometry. And the Moderns, by confounding both, have lost the Simplicity in which all the Elegancy of Geometry consists. Wherefore that is Arithmetically more simple which is determin'd by the more simple Æquations, but that is Geometrically more simple which is determin'd by the more simple drawing of Lines; and in Geometry, that ought to be reckon'd best which is Geometrically most simple. Wherefore, I ought not to be blamed, if, with that Prince of Mathematicians, *Archimedes*, and other Antients, I make Use of the Conchoid for the Construction of solid Problems. But if any one thinks otherwise, let him know, that I am here solicitous not for a Geometrical Construction but any one whatever, by which I may the nearest Way find the Root of the Æquation in Numbers, For the sake whereof I here premise this Lemmatical Problem.

To place the right Line BC of a given Length, so between two other given Lines AB, AC, that being produc'd, it shall pass through the given Point P.

IF the Line BC turn about the Pole P, and at the same time moves on its End C upon the right Line AC, its other End B shall describe the Conchoid of the Antients. Let this cut the Line AB in the Point B. Join PB, and its Part BC will be the right Line which was to be drawn. And, by the same Law, the Line BC may be drawn where, instead of AC, some Curve Line is made Use of. [*Vide Figure* 90.]

If any do not like this Construction by a Conchoid, another, done by a Conick Section, may be substituted in its room. From the Point P to the right Line AD, AE, draw PD, PE, making the Parallelogram $EADP$, and from the Points C and D to the right Lines AB let fall the Perpendiculars CF, DG, as also from the Point E to the right Line

Line AC, produc'd towards A, let fall the Perpendicular EH, and making $AD=a$, $PD=b$, $BC=c$, $AG=d$, $AB=x$, and $AC=y$, you'll have $AD:AG::AC:AF$, and consequently $AF=\frac{dy}{a}$. Moreover, you'll have $AB:AC::PD:CD$, or $x:y::b:a-y$. Therefore $by=ax-yx$, which is an Æquation expressive of an Hyperbola. And again, by the 13th of the 2d *Elem.* BCq will be $=ACq+ABq-2FAB$, that is, $cc=yy+xx-\frac{2dxy}{a}$. Both Sides of the former Æquation being multiply'd by $\frac{2d}{a}$, take them from both Sides of this, and there will remain $cc-\frac{2bdy}{a}=yy+xx-2dx$, an Æquation expressing a Circle, where x and y are at right Angles. Wherefore, if you make these two Lines an Hyperbola and a Circle, by the Help of these Æquations, by their Intersection you'll have x and y, or AB and AC, which determine the Position of the right Line BC. But those right Lines will be compounded after this Way.

Draw any two right Lines, KL equal to AD, and KM equal to PD, containing the right Angle MKL. Compleat the Parallelogram $KLMN$, and with the Asymptotes LN, MN, describe through the Point K the Hyperbola IKX.

On KM produc'd towards K, take KP equal to AG, and KQ equal to BC. And on KL produc'd towards K, take KR equal to AH, and RS equal to RQ. Compleat the Parallelogram $PKRT$, and from the Center T, at the Interval TS, describe a Circle. Let that cut the Hyperbola in the Point X. Let fall to KP the Perpendicular XY, and XY will be equal to AC, and KY equal to AB. Which two Lines, AC and AB, or one of them, with the Point P, determine the Position sought of the right Line BC. To demonstrate which Construction, and its Cases, according to the [different] Cases of the Problem, I shall not here insist. [*Vide Figure* 91.]

I say, by this Construction, if you think fit, you may solve the Problem. But this Solution is too compounded to serve for any [particular] Uses. It is only a Speculation, and Geometrical Speculations have just as much Elegancy as

Simplicity,

Simplicity, and deserve just so much Praise as they can promise Use. For which Reason, I prefer the Conchoid, as much the simpler, and not less Geometrical; and which is of especial Use in the Resolution of Æquations as by us propos'd. Premising therefore the preceding Lemma, we Geometrically construct Cubick and Biquadratick Problems [*as which may be reduc'd to Cubick ones*] as follows. [*Vide Figures 92 and 93.*]

Let there be propos'd the Cubick Æquation $x^3 * + qx + r = 0$, whose second Term is wanting, but the third is denoted under its Sign $+ q$, and the fourth by $+ r$. Draw any right Line, KA, which call n. On KA, produc'd on both Sides, take $KB = \frac{q}{n}$ to the same Side as KA, if q be positive, otherwise to the contrary Part. Bisect BA in C, and on K, as a Center with the Radius KC, describe the Circle CX, and in it accommodate the right Line CX equal to $\frac{r}{nn}$, producing it each Way. Join AX, which produce also both Ways; then between the Lines CX and AX inscribe EY of the same Length as CA, and which being produc'd, may pass through the Point K; then shall XY be the Root of the Æquation. [*Vide Figure 94.*] And of these Roots, those will be Affirmative which fall from X towards C, and those Negative which fall on the contrary Side, if it be $+ r$, but contrarily if it be $- r$.

Demonstration.

To demonstrate which, I premise these Lemma's.

LEMMA I. $YX : AK :: CX : KE$. Draw KF parallel to CX; then because of the similar Triangles ACX, AKF, and EYX, EKF, there is $AC : AK :: CX : KF$, and $YX : YE$, or $AC :: KF : KE$; and therefore by Equality $YX : AK :: CX : KE$. Q. E. D.

LEMMA II. $YX : AK :: CY : \overline{AK + KE}$. For by Composition of Proportion $YX : AK :: \overline{YX + CX}$ (i. e. CY) $: \overline{AK + KE}$. Q. E. D.

LEMMA

LEMMA III. $KE - BK : \Upsilon X :: \Upsilon X : AK$. For (by 12. *Elem.* 2.) $\Upsilon Kq - CKq = C\Upsilon q - C\Upsilon \times CX = C\Upsilon \times \Upsilon X$. That is, if the Theorem be resolv'd into Proportionals, $C\Upsilon : \Upsilon K - CK :: \Upsilon K + CK : \Upsilon X$. But $\Upsilon K - CK = \Upsilon K - \Upsilon E + CA - CK = KE - BK$. And $\Upsilon K + CK = \Upsilon K - \Upsilon E + CA + CK = KE + AK$. Wherefore $C\Upsilon : KE - BK :: KE + AK : \Upsilon X$. But by *Lemma* 2. $C\Upsilon : KE + AK :: \Upsilon X : AK$. Wherefore by Equality $\Upsilon X : KE - BK :: AK : \Upsilon X$; or $KE - BK : \Upsilon X :: \Upsilon X : AK$. Q. E. D.

These Things being premised, the Theorem will be thus demonstrated.

In the first *Lemma*, $\Upsilon X : AK :: CX : KE$, or $KE \times \Upsilon X = AK \times CX$; and in the third *Lemma* it was prov'd, that $KE - BK : \Upsilon X :: \Upsilon X : AK$. Wherefore, if the Terms of the first Ratio of the last Proportion be multiply'd by ΥX, it will be $KE \times \Upsilon X - BK \times \Upsilon X : X\Upsilon q :: \Upsilon X : AK$, that is, $AK \times CX - BK \times \Upsilon X : \Upsilon Xq :: \Upsilon X : AK$, and by multiplying the Extremes and Means into themselves, it will be $AKq \times XC - AK \times BK \times \Upsilon X = \Upsilon X\,cube$. Therefore for ΥX, AK, BK, and CX, re-substituting x, n, $\frac{q}{n}$, and $\frac{r}{nn}$, this Æquation will arise, *viz.* $r - qx = x^3$. Q. E. D. I need not stay to shew you the Variations of the Signs, for they will be determin'd according to the different Cases of the Problem.

Let then an Æquation be propos'd wanting the third Term, as $x^3 + pxx + r = 0$; in order to construct which, take n for any Number of equal Parts; take also, in any right Line, two Lengths $KA = \frac{r}{nn}$, and $KB = p$, and let them be taken the same Way if r and p have like Signs; but otherwise, take them towards contrary Sides. Bisect BA in C, and on K, as a Center, with the Radius KC, describe a Circle, into which accommodate $CX = n$, producing it both Ways. Join AX, produce it both Ways. Then, between the Lines CX and AX draw $E\Upsilon = CA$, so that it produc'd it may pass through the Point K; and KE will be the Root of the Æquation. And the Roots will be Affirmative, when the Point Υ falls on that Side of X which lies towards C; and Negative, when it falls on the contrary

 Side

Side of X, provided it be $+r$; but if it be $-r$, it will be the Reverse of this.

To demonstrate this Proposition, look back to the Figures and Lemma's of the former; and then you will find it thus.

By *Lemma* 1. $YX : AK :: CX : KE$, or $YX \times KE = AK \times CX$, and by *Lemma* 3, $KE - KB : YX :: YX : AK$, or, (taking KB towards contrary Parts) $KE + KB : YX :: YX : AK$, and therefore $KE + KB$ multiply'd by KE will be to $YX \times KE$: (or $AK \times CX$) :: $YX : AK$, or as $CX : KE$. Wherefore multiplying the Extreams and Means into themselves, KE *cube* $+ KB \times KEq = AK \times CXq$; and then for KE, KB, AK, and CK, restoring their Substitutes, you will find the last Æquation to be the same with what was propos'd, $x^3 + pxx = r$, or $x^3 + pxx \pm r = 0$.

Let an Æquation, having three Dimensions, and wanting no Term, be propos'd in this Form, $x^3 + pxx + qx + r = 0$, some of whose Roots shall be Affirmative, and some Negative.

And first suppose q a Negative Quantity, then in any right Line, as KB, let two Lengths be taken, as $KA = \frac{r}{q}$, and $KB = p$, and take them the same Way, if p and $\frac{r}{q}$ have contrary Signs; but if their Signs are alike, then take the Lengths contrary Ways from the Point K. Bisect AB in C, and there erect the Perpendicular CX equal to the Square Root of the Term q; then between the Lines AX and CX, produc'd infinitely both Ways, inscribe the right Line $EY = AC$, so that being produc'd, it may pass through K; so shall KE be the Root of the Æquation, which will be Affirmative when the Point X falls between A and E; but Negative when the Point E falls on that Side of the Point X which is towards A. [*Vide Figure* 95.]

If q had been an Affirmative Quantity, then in the Line KB you must have taken those two Lengths thus, *viz.* $KA = \sqrt{\frac{-r}{p}}$, and $KB = \frac{q}{KA}$, and the same Way from K, if $\sqrt{\frac{-r}{p}}$ and $\frac{q}{KA}$ have different Signs; but contrary Ways, if the Signs are of the same Nature. BA also must be be bisected in C; and there the Perpendicular CX erected equal to the Term p; and between the Lines AX and CX, infinitely drawn out both Ways, the right Line EY must also be inscribed equal to AC, and made to pass through the Point K, as before; then would XY be the Root of the Æquation; Negative when the Point X should fall between A and E, and Affirmative when the Point Y falls on the Side of the Point X towards C.

The Demonstration of the first Case.

By the first Lemma, KE was to CX as AK to YX, and (by Composition) so $KE + AK$, i. e. $KY + KC$ is to $CX + YX$, i. e. CY. But in the right-angled Triangle KCY, $YCq = YKq - KCq = \overline{KY + KC} \times \overline{KY - KC}$; and by resolving the equal Terms into Proportionals, $KY + KC$ is to CY as CY is to $KY - KC$; or $KE + AK$ is to CY as CY is to $EK - KB$. Wherefore since KE was to XC in this Proportion, by Duplication KEq will be to CXq as $KE + AK$ to $KE - KB$, and by multiplying the Extreams and Means by themselves $KE\,cube - KB \times KEq = CXq \times KE + CXq \times AK$. And by restoring the former Values $x^3 - pxx = qx + r$.

The Demonstration of the second Case.

By the first Lemma, KE is to CX as AK is to YX, then by multiplying the Extreams and Means by themselves, $KE \times YX = CX \times AK$. Therefore in the preceding Case, put $KE \times YX$ for $CX \times AK$, and it will be $KE\,cub. - KB \times KEq = CXq \times KE + CX \times KE \times YX$; and by dividing all by KE, there will be $KEq - KB \times KE = CXq + CX \times YX$; then multiplying all by AK, and you'll have $AK \times KEq - KB \times KA \times KE = AK \times CXq + AK \times CX \times YX$. And again, put $KE \times YX$ instead of its equal $CX \times AK$, then $AK \times KEq - AK \times KB \times KE = EK \times CX \times YX + KE \times YXq$; whence all being divided by KE there will arise $AK \times KE - AK \times KB = YX \times CX + YXq$; and when all are multiply'd by YX there will be $AK \times KE \times YX - AK \times KB \times YX = YXq \times CX + YX\,cube$. And instead of $KE \times YX$ in the first Term, put $CX \times AK$, and then $CX \times AKq - AK \times BK \times YX = CX \times$

 YXq

$YXq + YX\,cube$, or, which is the same Thing, $YX\,cube + CX \times YXq + AK \times KB \times YX - CX \times AKq = 0$. And by substituting for YX, CX, AK, and KB, their Values x, p, $\sqrt{\frac{-r}{p}}$, $q\sqrt{\frac{p}{-r}}$, this Æquation will come out, $x^3 + pxx + qx + r = 0$.

But these Æquations are also solv'd, by drawing a right Line from a given Point, in such a Manner that the Part of it, which is intercepted between another right Line and a Circle, both given in Position, may be of a given Length. [*Vide Figure 96.*]

For, let there be propos'd a Cubick Æquation $x^3 * + qx + r = 0$, whose second Term is wanting. Draw the right Line KA at Pleasure, which call n. In KA, produc'd both Ways, take $KB = \frac{q}{n}$ on the same Side of the Point K as the Point A is if q be Negative, if not, on the contrary. Bisect BA in C, and from the Center A, with the Distance AC, describe a Circle CX. To this inscribe the right Line $CX = \frac{r}{nn}$, and through the Points K, C, and X describe the Circle $KCXG$. Join AX, and produce it till it again cuts the Circle $KCXG$ last describ'd in the Point G. Lastly, between this Circle $KCXG$, and the right Line KC produc'd both Ways, inscribe the right Line $EY = AC$, so that EY produc'd pass through the Point G. And EG will be one of the Roots of the Æquation. But those Roots are Affirmative which fall in the greater Segment of the Circle KGC, and Negative which fall in the lesser KFC, if r is Negative, and the contrary will be when r is Affirmative.

In order to demonstrate this Construction, let us premise the following *Lemmata*.

LEMMA I. All Things being suppos'd as in the Construction, CE is to KA as $CE + CX$ is to AY, and as CX to KY.

For the right Line KG being drawn, AC is to AK as CX is to KG, because the Triangles ACX and AKG are Similar. The Triangles YEC, YKG are also Similar; for the Angle at Y is common to both Triangles, and the Angles G and C are in the same Segment KCG of the Circle $EGCK$, and

and therefore equal. Whence CE will be to EY as KG to KY, that is, CE to AC as KG to KY, because EY and AC were supposed equal. And by comparing this with the Proportionality above, it will follow by Equality of Proportion that CE is to KA as CX to KY, and alternately CE is to CX as KA to KY. Whence, by Composition, $CE+CX$ will be to CX as $KA+KY$ to KY, that is, AY to KY, and alternately $CE+CX$ is to AY as CX is to KY, that is, as CE to KA. Q. E. D.

LEMMA II. Let fall the Perpendicular CH upon the right Line GY, and the Rectangle $2HEY$ will be equal to the Rectangle $CE \times CX$.

For the Perpendicular GL being let fall upon the Line AY, the Triangles KGL, ECH have right Angles at L and H, and the Angles at K and E are in the same Segment CGK of the Circle $CKEG$, and are therefore equal; consequently the Triangles are Similar. And therefore KG is to KL as EC to EH. Moreover, AM being let fall from the Point A perpendicular to the Line KG, because AK is equal to AG, KG will be bisected in M; and the Triangles KAM and KGL are Similar, because the Angle at K is common, and the Angles at M and L are right ones; and therefore AK is to KM as KG is to KL. But as AK is to KM so is $2AK$ to $2KM$, or KG; (and because the Triangles AKG and ACX are Similar) so is $2AC$ to CX; also (because $AC=EY$) so is $2EY$ to CX. Therefore $2EY$ is to CX as KG to KL. But KG was to KL as EC to EH, therefore $2EY$ is to CX as EC to EH, and so the Rectangle $2HEY$ (by multiplying the Extreams and Means by themselves) is equal to $EC \times CX$. Q. E. D.

Here we took the Lines AK and AG equal. For the Rectangles CAK and XAG are equal (by *Cor.* to 36 *Prop.* of the 3d Book of *Euc.*) and therefore as CA is to XA so is AG to AK. But XA and CA are equal by Hypothesis; therefore $AG=AK$.

LEMMA III. All Things being as above, the three Lines BY, CE, KA are continual Proportionals.

For (by *Prop.* 12. *Book* 2. *Elem.*) $CYq=EYq+CEq+2EY \times EH$. And by taking EYq from both Sides, $CYq-EYq=CEq+2EY \times EH$. But $2EY \times EH=CE \times CX$ (by *Lem.* 2.) and by adding CEq to both Sides, $CEq+$

$2EY$

$2EY \times EH = CEq + CE \times CX$. Therefore $CYq - EYq = CEq + CE \times CX$, that is, $\overline{CY + EY} \times \overline{CY - EY} = CEq + CE \times CX$. And by resolving the equal Rectangles into proportional Sides, it will be as $CE + CX$ is to $CY + EY$, so is $CY - EY$ to CE. But the three Lines EY, CA, CB, are equal, and thence $CY + EY = CY + CA = AY$, and $CY - EY = CY - CB = BY$. Write AY for $CY + EY$, and BY for $CY - EY$, and it will be as $CE + CX$ is to YA so is BY to CE. But (by *Lem.* 1.) CE is to KA as $CE + CX$ is to AY, therefore CE is to KA as BY is to CE, that is, the three Lines BY, CE, and KA are continual Proportionals. Q. E. D.

Now, by the Help of these three Lemmas, we may demonstrate the Construction of the preceding Problem, thus: By *Lem.* 1. CE is to KA as CX is to KY, so $KA \times CX = CE \times KY$, and by dividing both Sides by CE, $\frac{KA \times CX}{CE} = KY$. To these equal Sides add BK, and $BK + \frac{KA \times CX}{CE} = BY$. Whence (by *Lem.* 3.) $BK + \frac{KA \times CX}{CE}$ is to CE as CE is to KA, and thence, by multiplying the Extreams and Means by themselves, $CEq = BK \times KA + \frac{KAq \times CX}{CE}$, and both Sides being multiply'd by CE, $CE\ cub. = KB \times KA \times CE + KAq \times CX$. CE was called x, the Root of the Æquation $KA = n$, $KB = \frac{q}{n}$, and $CX = \frac{r}{nn}$. These being substituted instead of CE, KA, KB, and CX, there will arise this Æquation, $x^3 = qx + r$, or $x^3 - qx - r = 0$; when q and r are Negatives, KA and KB having been taken on the same Side of the Point K, and the Affirmative Root being in the greater Segment CGK. This is one Case of the Construction to be demonstrated. Draw KB on the contrary Side, that is, let its Sign be changed, or the Sign of $\frac{q}{n}$, or, which is the same Thing, the Sign of the Term q, and there will be had the Construction of the Æquation $x^3 + qx - r = 0$. Which is the other Case. In these Cases CX, and the Affirmative Root CE, fall towards the same Parts of the Line AK. Let CX and

and the Negative Root fall towards the ſame Parts when the Sign of CX, or $\frac{r}{nn}$, or (which is the ſame Thing) r is changed; and this will be the third Caſe $x^3 + qx + r = 0$, where all the Roots are Negative. And again, when the Sign of KB, or $\frac{q}{n}$, or only q, is changed, it will be the fourth Caſe $x^3 - qx + r = 0$. The Conſtructions of all theſe Caſes may be eaſily run through, and particularly demonſtrated after the ſame Manner as the firſt was; and with the ſame Words, by changing only the Situation of the Lines.

Now let the Cubick Æquation $x^3 + pxx * + r = 0$, whoſe third Term is wanting, be to be conſtructed.

In the ſame Figure n being taken of any Length, take in any infinite right Line AY, KA, and $KB = \frac{r}{nn}$, and p, and take them on the ſame Side of the Point K, if the Signs of the Terms p and r are the ſame, otherwiſe on contrary Sides. Biſect BA in C, and from the Center K with the Diſtance KC deſcribe the Circle CXG. And to it inſcribe the right Line CX equal to n the aſſumed Length. Join AX and produce it to G, ſo that AG may be equal to AK, and through the Points K, C, X, G deſcribe a Circle. And, laſtly, between this Circle and the right Line KC, produc'd both Ways, draw the right Line $EY = AC$, ſo that being produced it may paſs through the Point G; then the right Line KY being produc'd, will be one of the Roots of the Æquation. And thoſe Roots are Affirmative which fall on that Side of the Point K on which the Point A is on, if r is Affirmative; but if r is Negative, then the Affirmative Roots fall on the contrary Side. And if the Affirmative fall on one Side, the Negative fall on the other.

This Conſtruction is demonſtrated by the Help of the three laſt *Lemma's* after this Manner:

By the third *Lemma*, BY, CE, KA are continual Proportionals; and by *Lemma* 1. as CE is to KA ſo is CX to KY. Therefore BY is to CE as CX to KY. $BY = KY - KB$. Therefore $KY - KB$ is to CE as CX is to KY. But as $KY - KB$ is to CE ſo is $\overline{KY - KB} \times KY$ to $CE \times KY$, by *Prop.* 1. *Book* 6. *Euc.* and becauſe of the Proportionals CE to KA as CX to KY is $CE \times KY = KA \times CX$.

There-

Therefore $\overline{KY - KB} \times KY$ is to $KA \times CX$ (as $KY - KB$ to CE, that is, as CX to KY. And by multiplying the Extreams and Means by themselves $\overline{KY - KB} \times KYq = KA \times CXq$; that is, $KY cub. - KB \times KYq = KA \times CXq$. But in the Construction KY was x the Root of the Æquation, KB was put $= p$, $KA = \frac{r}{nn}$, and $CX = n$. Write therefore x, p, $\frac{r}{nn}$, and n for KY, KB, KA, and CX respectively, $x^3 - pxx$ will be equal to r, or $x^3 - pxx - r = 0$.

This Construction may be resolv'd into four Cases of Æquations, $x^3 - pxx - r = 0$, $x^3 - pxx + r = 0$, $x^3 + pxx - r = 0$, and $x^3 + pxx + r = 0$. The first Case I have already demonstrated; the rest are demonstrated with the same Words, only changing the Situation of the Lines. To wit, as in taking KA and KB on the same Side of the Point K, and the Affirmative Root KY on the contrary Side, has already produc'd $KY cub. - KB \times KYq = KA \times CXq$, and thence $x^3 - pxx - r = 0$; so by taking KB on the other Side the Point K, it will produce, by the like Reasoning, $KY cub. + KYq \times KB = KA \times CXq$, and thence $x^3 + pxx - r = 0$. And in these two Cases, if the Situation of the Affirmative Root KY be changed, by taking it on the other Side of the Point K, by a like Series of Arguments, it will fall into the other two Cases, $KY cub. + KB \times KYq = - KA \times CXq$, or $x^3 + pxx + r = 0$, and $KY cub. - KB \times KYq = - KA \times CXq$, or $x^3 - pxx + r = 0$. Which were all the Cases to be demonstrated.

Now let this Cubick Æquation $x^3 + pxx + qx + r = 0$ be propos'd, wanting no Term (unless perhaps the third). Which is constructed after this Manner: [*Vide Figures 97 and 98.*]

Take n at Pleasure. Draw any right Line $GC = \frac{n}{2}$, and at the Point G erect a Perpendicular $GD = \sqrt{\frac{r}{p}}$, and if the Terms p and r have contrary Signs, from the Center C, with the Interval CD describe a Circle PBE. If they have the same Signs, from the Center D, with the Space GC, describe an occult Circle, cutting the right Line GA in H;

H; then from the Center C, with the Diſtance GH, deſcribe the Circle PBE. Then make $GA = -\frac{q}{n} - \frac{r}{np}$ on the ſame Side the Point G that C is on, if now the Quantity $-\frac{q}{n} - \frac{r}{np}$ (the Signs of the Terms p, q, r in the Æquation to be conſtructed being well obſerv'd) ſhould come out Affirmative; otherwiſe, draw GA on the other Side of the Point G, and at the Point A erect the Perpendicular AY, between which and the Circle PBE already deſcrib'd, draw the right Line EY equal to p, ſo that being produc'd, it may paſs through the Point G; which being done, the Line EG will be one of the Roots of the Æquation to be conſtructed. Thoſe Roots are Affirmative where the Point E falls between the Points G and Y, and Negative, where the Point E falls without, if p is Affirmative; and the contrary, if Negative.

In order to demonſtrate this Conſtruction, let us premiſe the following Lemmas.

LEMMA I. Let EF be let fall perpendicular to AG, and the right Line EC be drawn; $EGq + GCq = ECq + 2CGF$. For (by *Prop.* 12. *Book* 2. *Elem.*) $EGq = ECq + GCq + 2GCF$. Let GCq be added on both Sides, and $EGq + GCq = ECq + 2GCq + 2GCF$. But $2GCq + 2GCF = 2GC \times \overline{GC + CF} = 2CGF$. Therefore $EGq + GCq = ECq + 2CGF$. Q. E. D.

LEMMA II. In the firſt Caſe of the Conſtruction, where the Circle PBE paſſes through the Point D, $GEq - GDq = 2CGF$. For by the firſt Lemma $EGq + GCq = ECq + 2CGF$, and by taking CGq from both Sides, $EGq = ECq - GCq + 2CGF$. But $ECq - GCq = CDq - GCq = GDq$. Therefore $EGq = GDq + 2CGF$, and by taking GDq from both Sides, $EGq - GDq = 2CGF$. Q. E. D.

LEMMA III. In the ſecond Caſe of the Conſtruction, where the Circle PCD does not paſs through the Point D, $EGq + GDq = 2CGF$. For, by the firſt Lemma, $EGq + GCq = ECq + 2CGF$. Take ECq from both Sides, and $EGq + GCq - ECq = 2CGF$. But $GC = DH$, and EC

 $= CP$

$= CP = GH$. Therefore $GCq - ECq = DHq - GHq = GDq$, and so $EGq + GDq = 2CGF$. Q. E. D.

Lemma IV. $GY \times 2CGF = 2CG \times AGE$. For, by reason of the similar Triangles GEF and GYA, as GF is to GE so is AG to GY, that is, (by *Prop.* 1. *Book* 6. *Elem.*) as $2CG \times AG$ is to $2CG \times GY$. Let the Extreams and Means be multiply'd by themselves, and $2CG \times GY \times GF = 2CG \times AG \times GE$. Q. E. D.

Now, by the Help of these Lemmas, the Construction of the Problem may be thus demonstrated.

In the first Case, $EGq - GDq = 2CGF$ (by Lemma 2.) and by multiplying all by GY, $EGq \times GY - GDq \times GY = 2CGF \times GY =$ (by Lemma 4.) $2CG \times AGE$. Instead of GY write $EG + EY$, and $EG\,cub. + EY \times EGq - GDq \times EG - GDq \times EY = 2CGA \times EG$, or $EG\,cub. + EY \times EGq + \overline{- GDq - 2CGA} \times EG - GDq \times EY = 0$.

In the second Case, $EGq + GDq = 2CGF$ (by Lemma 3.) and by multiplying all by GY, $EGq \times GY + GDq \times GY = 2CGF \times GY = 2CG \times AGE$, by Lemma 4. Instead of GY write $EG + EY$, and $EG\,cub. + EY \times EGq + GDq + EG + GDq \times EY = 2CGA \times EG$, or $EG\,cub. + EY \times EGq + \overline{GDq - 2CGA} \times EG + GDq \times EY = 0$.

But the Root of the Æquation $EG = x$, $GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2CG = n$, and $GA = -\frac{q}{n} - \frac{r}{np}$, that is, in the first Case, where the Signs of the Terms p and r are different; but in the second Case, where the Sign of one of the two, p or r, is changed, there is $-\frac{q}{n} + \frac{r}{np} = GA$. Let therefore EG be put $= x$, $GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2CG = n$, and $GA = -\frac{q}{n} \mp \frac{r}{np}$, and in the first Case it will be $x^3 + px^2 + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x - r = 0$; that is, $x^3 + px^2 + qx - r = 0$; but in the second Case, $x^3 + pxx + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x + r = 0$, that is, $x^3 + px^2 + qx + r = 0$.

$=0$. Therefore in both Cases EG is the true Value of the Root x. Q. E. D.

But either Case may be distinguish'd into its several Particulars; as the former into these, $x^3+px^2+qx-r=0$, $x^3+px^2-qx-r=0$, $x^3-px^2+qx+r=0$, $x^3-px^2-qx+r=0$, $x^3+px^2-r=0$, and $x^3-px^2+r=0$; the latter into these, $x^3+px^2+qx+r=0$, $x^3+px^2-qx+r=0$, $x^3-px^2+qx-r=0$, $x^3-px^2-qx-r=0$, $x^3+px^2+r=0$, and $x^3-px^2-r=0$. The Demonstration of all which Cases may be carry'd on in the same Words with the two already demonstrated, by only changing the Situation of the Lines.

These are the chief Constructions of Problems, by inscribing a right Line given in Length so between a Circle and a right Line given in Position, that the inscrib'd right Line produc'd may pass through a given Point. And such a right Line may be inscrib'd by describing a Conchoid, of which let that Point, through which the right Line given ought to pass, be the Pole, the other right Line given in Position; the Ruler or Asymptote, and the Interval, the Length of the inscrib'd Line. For this Conchoid will cut the Circle in the Point E, through which the right Line to be inscrib'd must be drawn. But it will be sufficient in Practice to draw the right Line between a Circle and a right Line given in Position by any Mechanick Method.

But in these Constructions observe, that the Quantity n is undetermin'd and left to be taken at Pleasure, that the Construction may be more conveniently fitted to particular Problems. We shall give Examples of this in finding two mean Proportionals, and in trisecting an Angle.

Let x and y be two mean Proportionals to be found between a and b. Because a, x, y, b are continual Proportionals, a^2 will be to x^2 as x to b, therefore $x^3=baa$, or $x^3-aab=0$. Here the Terms p and q of the Æquation are wanting, and $-aab$ is in the room of the Term r; therefore in the first Form of the Constructions, where the right Line EY tending to the given Point K, is drawn between other two right Lines, EX and YC, given in Position, and suppose the right Line $CX=\frac{r}{nn}=\frac{-aab}{nn}$, let n be taken equal to a, and then CX will be $=-b$. From whence the like Construction comes out. [*Vide Figure* 99.]

 I draw

I draw any Line, $KA = a$, and bisect it in C, and from the Center , with the Distance AC, describe the Circle CX, to which I inscribe the right Line $CX = b$, and between AX and CX infinitely produc'd, I so inscribe $EY = CA$, that EY being produc'd, may pass through the Point K. So KA, XY, KE, CX will be continual Proportionals, that is, XY and KE two mean Proportionals between a and b. This Construction is known. [*Vide Figure* 100.]

But in the other Form of the Constructions, where the right Line EY converging to the given Point G is inscrib'd between the Circle $GECX$ and the right Line AK, and $CX = \frac{r}{nn}$, that is, (in this Problem) $= \frac{-aab}{nn}$, I put, as before, $n = a$, and then CX will be $= b$, and the rest are done as follows. [*Vide Figure* 101.]

I draw any right Line $KA = a$, and bisect it in C, and from the Center A, with the Distance AK, I describe the Circle KG, to which I inscribe the right Line $KG = 2b$, constituting the *Isosceles* Triangle AKG. Then, through the Points C, K, G I describe the Circle, between the Circumference of which and the right Line AK produc'd, I inscribe the right Line $EY = CK$ tending to the Point G. Which being done, AK, EC, KY, $\frac{1}{2}KG$ are continual Proportionals, that is, EC and KY are two mean Proportionals between the given Quantities a and b.

Let there be an Angle to be divided into three equal Parts; [*Vide Figure* 102.] and let that Angle be ACB, and the Parts thereof to be found be ACD, ECD, and ECB; from the Center C, with the Distance CA, let the Circle $ADEB$ be describ'd, cutting the right Lines CA, CD, CE, CB in A, D, E, B. Let AD, DE, EB be join'd, and AB cutting the right Lines CD, CE at F and H, and let DG, meeting AB in G, be drawn parallel to CE. Because the Triangles CAD, ADF, and DFG are Similar, CA, AD, DF, and FG are continual Proportionals. Therefore if $AC = a$, and $AD = x$, DF will be equal to $\frac{xx}{a}$, and $FG = \frac{x^3}{aa}$. And $AB = BH + HG + FA - GF = 3AD - GF = 3x - \frac{x^3}{aa}$. Let $AB = b$, then $b = 3x - \frac{x^3}{aa}$, or $x^3 - 3aax + aab = 0$. Here p, the se-

cond

cond Term of the Æquation, is wanting, and instead of q and r we have $-3aa$ and aab. Therefore in the first Form of the Constructions, where p was $=0$, $KA=n$, $KB=\frac{q}{n}$, and $CX=\frac{r}{nn}$, that is, in this Problem, $KB=-\frac{3aa}{n}$, and $CX=\frac{aab}{nn}$, that these Quantities may come out as simple as possible, I put $n=a$, and so $KB=-3a$, and $CX=b$. Whence this Construction of the Problem comes out.

Draw any Line, $KA=a$, and on the contrary Side make $KB=3a$. [*Vide Figure* 103.] Bisect BA in C, and from the Center K, with the Distance KC, describe a Circle, to which inscribe the right Line $CX=b$, and the right Line AX being drawn between that infinitely produc'd and the right Line CX, inscribe the right Line $EY=AC$, and so that it being produc'd, will pass through the Point K. So XY will be $=x$. But (see the last Figure) because the Circle $ADEB=$ Circle CXA, and the Subtense $AB=$ Subtense CX, and the Parts of the Subtenses BH and XY are equal; the Angles ACB, and CKX will be equal, as also BCH, XKY; and so the Angle XKY will be one third Part of the Angle CKX. Therefore the third Part XKY of any given Angle CKX is found by inscribing the right Line $EY=AC$, the Diameter of the Circle between the Chords CX and AX infinitely produc'd, and converging at K the Center of the Circle.

Hence, if from K, the Center of the Circle, you let fall the Perpendicular KH upon the Chord CX, the Angle HKY will be one third Part of the Angle HKX; so that if any Angle HKX were given, the third Part thereof HKY may be found by letting fall from any Point X of any Side KX, the Line HX perpendicular to the other Side HK, and by drawing XE parallel to HK, and by inscribing the right Line $YE=2XK$ between XH and XE, so that it being produc'd may pass through the Point K. Or thus. [*Vide Figure* 104.]

Let any Angle AXK be given. To one of its Sides AX raise a Perpendicular XH, and from any Point K of the other Side XK let there be drawn the Line KE, the Part of which EY (lying between the Side AX produc'd, and the Perpendicular XH) is double the Side XK, and the Angle KEA will be one third of the given Angle AXK.

Again,

Again, the Perpendicular EZ being rais'd, and KF being drawn, whose Part ZF, between EF and EZ, let be double to KE, and the Angle KFA will be one third of the Angle KEA; and so you may go on by a continual Trisection of an Angle *ad infinitum*. This Method is in the 32d *Prop.* of the 4th Book of *Pappus*.

If you would trisect an Angle by the other Form of Constructions, where the right Line is to be inscrib'd between another right Line and a Circle, here also will $KB = \frac{q}{n}$, and $CX = \frac{r}{nn}$, that is, in the Problem we are now about, $KB = \frac{-3aa}{n}$, and $CX = \frac{aab}{nn}$; and so by putting $n = a$, KB will be $= -3a$, and $CX = b$. Whence this Construction comes out.

From any Point K let there be drawn two right Lines towards the same Way, $KA = a$, and $KB = 3a$. [*Vide Figure* 105.] Bisect AB in C, and from the Center A with the Distance AC describe a Circle. To which inscribe the right Line $CX = b$. Join AX, and produce it till it cuts the Circle again in G. Then between this Circle and the right Line AC, infinitely produc'd, inscribe the Line $EY = AC$, and passing through the Point G; and the right Line EC being drawn, will be equal to x the Quantity sought, by which the third Part of the given Angle will be subtended.

This Construction arises from the Form above; which, however, comes out better thus: Because the Circles $ADEB$ and KXG are equal, and also the Subtenses CX and AB, the Angles CAX, or KAG, and ACB are equal, therefore CE is the Subtense of one third Part of the Angle KAG. Whence in any given Angle KAG, that its third Part CAE may be found, inscribe the right Line EY equal to the Semi-Diameter AG of the Circle KCG, between the Circle and the Side KA, of the Angle, infinitely produc'd, and tending to the Point G. Thus *Archimedes*, in *Lemma* 8. taught to trisect an Angle. The same Constructions may be more easily explain'd than I have done here; but in these I would show how, from the general Constructions of Problems I have already explain'd, we may derive the most simple Constructions of particular Problems.

Besides

Tab: VII: p: 246.

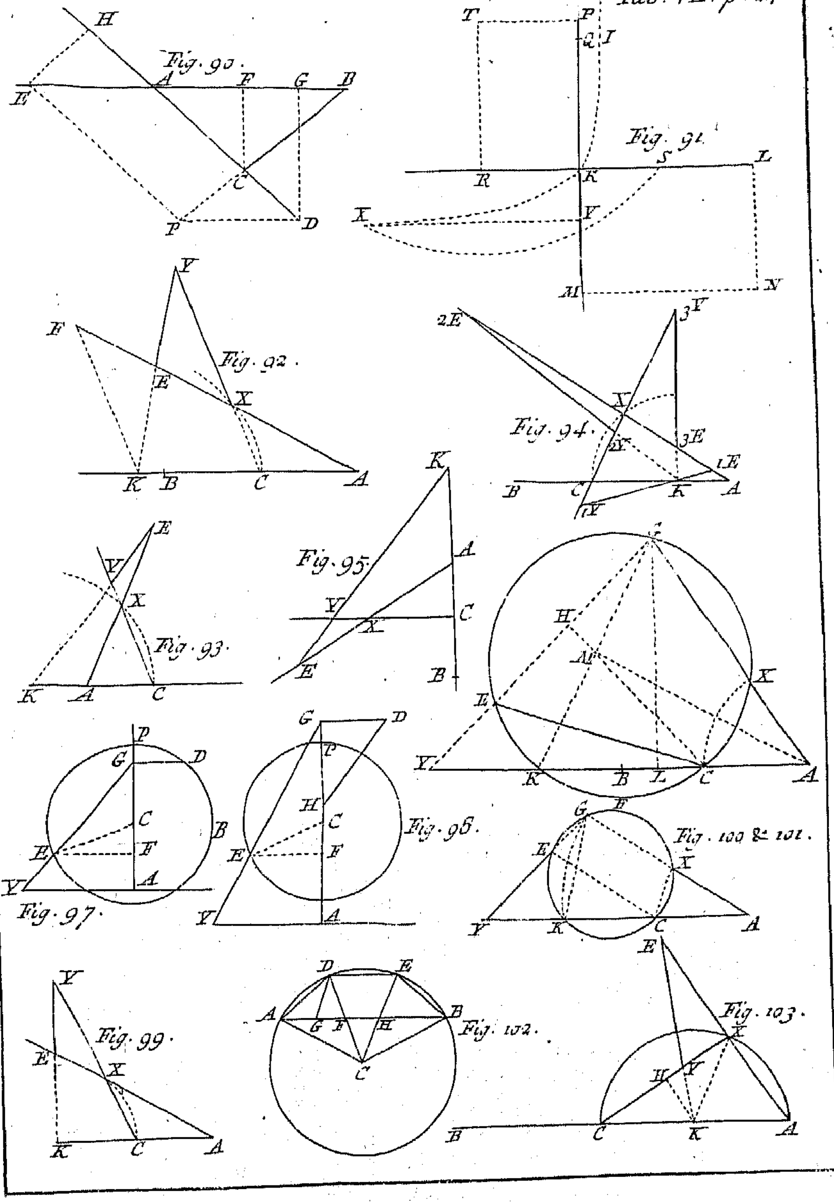
Fig. 90.
Fig. 91.
Fig. 92.
Fig. 93.
Fig. 94.
Fig. 95.
Fig. 96.
Fig. 97.
Fig. 99.
Fig. 100 & 101.
Fig. 102.
Fig. 103.

Besides the Constructions here set down, we might add many more. [*Vide Figure* 106.] As if there were two mean Proportionals to be found between a and b. Draw any right Line $AK = b$, and perpendicular to it $AB = a$. Bisect AK in I, and in AK put AH equal to the Subtense BI; and also in the Line AB produc'd, $AC =$ Subtense BH. Then in the Line AK on the other Side of the Point A take AD of any Length and DE equal to it, and from the Centers D and E, with the Distances DB and EC, describe two Circles, BF and CG, and between them draw the right Line FG equal to the right Line AI, and converging at the Point A, and AF will be the first of the two mean Proportionals that were to be found.

The Ancients taught how to find two mean Proportionals by the Cissoid; but no Body that I know of hath given a good manual Description of this Curve. [*Vide Figure* 107.] Let AG be the Diameter, and F the Center of a Circle to which the Cissoid belongs. At the Point F let the Perpendicular FD be erected, and produc'd *in infinitum*. And let FG be produc'd to P, that FP may be equal to the Diameter of the Circle. Let the Ruler PED be moved, so that the Leg EP may always pass through the Point P, and the other Leg ED must be equal to the Diameter AG, or FP, with its End D, always moving in the Line FD; and the middle Point C of this Leg will describe the Cissoid GCK which was desired, as has been already shewn. Wherefore, if between any two Quantities, a and b, there be two mean Proportionals to be found: Take $AM = a$, raise the Perpendicular $MN = b$. Join AN, and move the Ruler PED, as was just now shewn, until its Point C fall upon the right Line AN. Then let fall CB perpendicular to AP, take t to BH, and v to BG, as MN is to BC, and because AB, BH, BG, BC are continual Proportionals, a, t, v, b will also be continual Proportionals.

By the Application of such a Ruler other solid Problems may be constructed.

Let there be proposed the Cubick Æquation $x^3\ pxx - qx + r = 0$; where q is always Negative, r Affirmative, and p of any Sign. Make $AG = \frac{r}{q}$, and bisect it in F, and take $FR = \frac{p}{2}$, and that towards A if p is Affirmative, if not towards P. Moreover, make $AB = \sqrt{q}$, and erect the

Perpen-

Perpendiculars FD and BC. And in the Leg ED of the Ruler, take $ED = AG$ and $EC = AR$; then let the Leg of the Ruler be apply'd to the Scheme; so that the Point D may touch the Line FD, and the Point C the right Line BC, and BC will be the Root of the Æquation sought, $= x$.

Thus far, I think, I have expounded the Construction of solid Problems by Operations whose manual Practice is most simple and expeditious. So the Antients, after they had obtain'd a Method of solving these Problems by a Composition of solid Places, thinking the Constructions by the Conick Sections useless, by reason of the Difficulty of describing them, sought easier Constructions by the Conchoid, Cissoid, the Extension of Threads, and by any Mechanick Application of Figures. Since useful Things, though Mechanical, are justly preferable to useless Speculations in Geometry, as we learn from *Pappus*. So the great *Archimedes* himself neglected the Trisection of an Angle by the Conick Sections, which had been handled by other Geometricians before him, and taught how to trisect an Angle in his Lemma's as we have already explain'd. If the Antients had rather construct Problems by Figures not receiv'd in Geometry in that Time, how much more ought these Figures now to be preferr'd which are receiv'd by many into Geometry as well as the Conick Sections.

However, I don't agree to this new Sort of Geometricians, who receive all Figures into Geometry. Their Rule of admitting all Lines to the Construction of Problems in that Order in which the Æquations, whereby the Lines are defin'd, ascend to the Number of Dimensions, is arbitrary and has no Foundation in Geometry. Nay, it is false; for according to this Rule, the Circle should be joined with the Conick Sections, but all Geometers join it with the right Line; and this being an inconstant Rule, takes away the Foundation of admitting into Geometry all Analytick Lines in a certain Order. In my Judgment, no Lines ought to be admitted into plain Geometry besides the right Line and the Circle. Unless some Distinction of Lines might be first invented, by which a circular Line might be joined with a right Line, and separated from all the rest. But truly plain Geometry is not to be augmented by the Number of Lines. For all Figures are plain that are admitted into plain Geometry, that is, those which the Geometers postulate to be described *in plano*. And every plain Problem is that which may be constructed by plain Figures. So

there-

therefore admitting the Conick Sections and other Figures more compounded into plain Geometry, all the solid and more than solid Problems that can be constructed by these Figures will become plane. But all plane Problems are of the same Order. A right Line Analytically is more simple than a Circle; nevertheless, Problems which are constructed by right Lines alone, and those that are constructed by Circles, are of the same Order. These Things being postulated, a Circle is reduc'd to the same Order with a right Line. And much more the Ellipse, which differs much less from a Circle than a Circle from a right Line, by postulating the right Description thereof *in plano*, will be reduc'd to the same Order with the Circle. If any, in considering the Ellipse, should fall upon some solid Problem, and should construct it by the Help of the same Ellipse, and a Circle: This would be counted a plane Problem, because the Ellipse was suppos'd to be describ'd *in plano*, and every Construction besides will be solv'd by the Description of the Circle only. Wherefore, for the same Reason, every plane Problem whatever may be constructed by a given Ellipse.

For Example, [*Vide Figure* 108.] If the Center O of the given Ellipse $ADFG$ be requir'd, I would draw the Parallels AB, CD meeting the Ellipse in A, B, C, D; and also two other Parallels EF, GH meeting the Ellipse in E, F, G, H, and I would bisect them in I, K, L, M, and produce IK, LM, till they meet in O. This is a real Construction of a plane Problem by an Ellipse. There is no Reason that an Ellipse must be Analytically defin'd by an Æquation of two Dimensions. Nor that it should be generated Geometrically by the Section of a solid Figure. The Hypothesis, only considering it as already describ'd *in plano*, reduces all solid Problems constructed by it to the Order of plane ones, and concludes, that all plane ones may be rightly constructed by it. And this is the State of the *Postulate*. But perhaps, by the Power of Postulates it is lawful to mix that which is now done, and that which is given. Therefore let this be a Postulate to describe an Ellipse *in plano*, and then all those Problems that can be constructed by an Ellipse, may be reduc'd to the Order of plane ones, and all plane Problems may be constructed by the Ellipse.

It is necessary therefore that either plane and solid Problems be confused among one another, or that all Lines be flung out of plane Geometry, besides the right Line and the

Circle, unlefs it happens that fometime fome other is given in the State of conftructing fome Problem. But certainly none will permit the Orders of Problems to be confufed. Therefore the Conick Sections and all other Figures muft be caft out of plane Geometry, except the right Line and the Circle, and thofe which happen to be given in the State of the Problems. Therefore all thefe Defcriptions of the Conicks *in plano*, which the Moderns are fo fond of, are foreign to Geometry. Neverthelefs, the Conick Sections ought not to be flung out of Geometry. They indeed are not defcribed Geometrically *in plano*, but are generated in the plane Superficies of a geometrical Solid. A Cone is conftituted geometrically, and cut by a Geometrical Plane. Such a Segment of a Cone is a Geometrical Figure, and has the fame Place in folid Geometry, as the Segment of a Circle has in Plane, and for this Reafon its Bafe, which they call a Conick Section, is a Geometrical Figure. Therefore a Conick Section hath a Place in Geometry fo far as the Superficies is of a Geometrical Solid; but is Geometrical for no other Reafon than that it is generated by the Section of a Solid, and therefore was not in former Times admitted only into folid Geometry. But fuch a Generation is difficult, and generally ufelefs in Practice, to which Geometry ought to be moft ferviceable. Therefore the Antients betook themfelves to various Mechanical Defcriptions of Figures *in plano*. And we, after their Example, have handled in the preceding Conftructions. Let thefe Conftructions be Mechanical; and fo the Conftructions by Conick Sections defcrib'd *in plano* be Mechanical. Let the Conftructions by Conick Sections given be Geometrical; and fo the Conftructions by any other given Figures are Geometrical, and of the fame Order with the Conftructions of plane Problems. There is no Reafon that the Conick Sections fhould be preferr'd in Geometry before any other Figures, unlefs fo far as they are deriv'd from the Section of a Cone; they being generally unferviceable in Practice in the Solution of Problems. But leaft I fhould altogether neglect Conftructions by the Conick Sections, it will be proper to fay fomething concerning them, in which alfo we will confider fome commodious manual Defcription.

The Ellipfe is the moft fimple of the Conick Sections, moft known, and neareft of Kin to a Circle, and eafieft defcrib'd by the Hand *in plano*. Though many prefer the

Para-

Parabola before it, for the Simplicity of the Æquation by which it is expreſs'd. But by this Reaſon the Parabola ought to be preferr'd before the Circle it ſelf, which it never is. Therefore the reaſoning from the Simplicity of the Æquation will not hold. The modern Geometers are too fond of the Speculation of Æquations. The Simplicity of theſe is of an Analytick Conſideration. We treat of Compoſition, and Laws are not given to Compoſition from Analyſis; Analyſis does lead to Compoſition: But it is not true Compoſition before its freed from Analyſis. If there be never ſo little Analyſis in Compoſition, that Compoſition is not yet true. Compoſition in it ſelf is perfect, and far from a Mixture of Analytick Speculations. The Simplicity of Figures depend upon the Simplicity of their Geneſis and Ideas, and an Æquation is nothing elſe than a Deſcription (either Geometrical or Mechanical) by which a Figure is generated and rendered more eaſy to the Conception. Therefore we give the Ellipſe the firſt Place, and ſhall now ſhow how to conſtruct Æquations by it.

Let there be any Cubick Æquation propos'd, $x^3 = pxx + qx + r$, where p, q, and r ſignify given Co efficients of the Terms of the Æquations, with their Signs $+$ and $-$, and either of the Terms p and q, or both of them, may be wanting. For ſo we ſhall exhibit the Conſtructions of all Cubick Æquations in one Operation, which follows:

From the Point B in any given right Line, take any two right Lines, BC and BE, on the ſame Side the Point B, and alſo BD, ſo that it may be a mean Proportional between them. [*Vide Figure* 109] And call BC, n, in the ſame right Line alſo take $BA = \frac{q}{n}$, and that towards the Point C, if $-q$, if not, the contrary Way. At the Point A erect a Perpendicular, and in it take $AF = p$, $FG = AF$, $FI = \frac{r}{nn}$, and FH to FI as BC is to BE. But FH and FI are to be taken on the ſame Side of the Point F towards G, if the Terms p and r have the ſame Signs; and if they have not the ſame Signs, towards the Point A. Let the Parallelograms $IACK$ and $HAEL$ be compleated, and from the Center K, with the Diſtance KG, let a Circle be deſcrib'd. Then in the Line HL let there be taken HR on either Side the Point H, which let be to HL as

 BD

BD to BE; let GR be drawn, cutting EL in S, and let the Line GRS be moved with its Point R falling on the Line HL, and the Point S upon the Line EL, until the Point G in describing the Ellipse, meet the Circle, as is to be seen in the Position of $\gamma\rho\sigma$. For half the Perpendicular γX let fall, from γ the Point of meeting, to AE will be the Root of the Æquation. But G or γ is the End of the Rule GRS, or $\gamma\rho\sigma$, meeting the Circle in as many Points as there are possible Roots. And those Roots are Affirmative which fall towards the same Parts of the Line EA, as the Line FI drawn from the Point F does, and those are Negative which fall towards the contrary Parts of the Line AE if r is Affirmative; and contrarily if r is Negative.

But this Construction is demonstrated by the Help of the following Lemma's.

LEMMA I. All being suppos'd as in the Construction, $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + 2AG \times FI$.

For from the Nature of the Circle, $K\gamma q - CXq = \overline{\gamma X - AI}|^2$. But $K\gamma q = GIq + ACq$, and $CXq = \overline{AX - AC}|^2$, that is, $= AXq - 2CAX + ACq$, and so their Difference $GIq + 2CAX - AXq = \overline{\gamma X - AI}|^2 = \gamma Xq - 2AI \times \gamma X + AIq$. Subtract GIq from both, and there will remain $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + AIq - GIq$. But (by *Prop.* 4. *Book* 2. *Elem.*) $AIq = AGq + 2AGI + GIq$, and so $AIq - GIq = AGq + 2AGI$, that is, $= 2AG \times \overline{\tfrac{1}{2}AG + GI}$, or $= 2AG \times FI$, and thence $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + 2AG \times FI$. Q. E. D.

LEMMA II. All Things being constructed as above $2EAX - AXq = \frac{FI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AG \times FI$.

For it is known, that the Point γ, by the Motion of the Ruler $\gamma\rho\sigma$ assign'd above, describes an Ellipse, the Center whereof is L, and the two Axis coincide with the two right Lines LE and LH, of which that which is in LE $= 2\gamma\rho$, or $= 2GR$, and the other which is in $LH = 2\gamma\sigma$, or $= 2GS$. And the Ratio of these to one another is the same as that of the Line HK to the Line HL, or of the Line BD to the Line BE. Therefore the *Latus Transver-*

$$2CE \times AX = \frac{HI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AI \times X\gamma.$$

Let both Sides be multiply'd by FH, and $2FH \times CE \times AX = HI \times X\gamma q - 2FI \times AH \times X\gamma + 2AI \times FH \times X\gamma$. But $AI = HI + AH$, and so $2FI \times AH - 2FH \times AI = 2FI \times AH - 2FHA - 2FHI$. But $2FI \times HA - 2FHA = 2AHI$, and $2AHI - 2FHI = 2HI \times AF$. Therefore $2FI \times AH - 2FH \times AI = 2HI \times AF$, and so $2FH \times CE \times AX = HI \times X\gamma q - 2HI \times AF \times X\gamma$. And thence as HI is to FH, so is $2CE \times AX$ to $X\gamma q - 2AF \times X\gamma$. But by Construction HI is to FH as CE is to BC, and so as $2CE \times AX$ is to $2BC \times AX$, and thence $2BC \times AX = X\gamma q - 2AF \times X\gamma$, (by *Prop. 9. Book 5. Elem.*) But because the Rectangles are equal, the Sides are proportional, AX to $X\gamma - 2AF$, (that is, $X\gamma - AG$) as $X\gamma$ is to $2BC$. Q. E. D.

LEMMA IV. The same Things being still suppos'd, $2FI$ is to $AX - 2AB$ as $X\gamma$ is to $2BC$.

For if from the Equals in the third Lemma, to wit, $2BC \times AX = X\gamma q - 2AF \times X\gamma$, the Equals in the first Lemma be subtracted, there will remain $-2AB \times AX + AXq = 2FI \times X\gamma - 2AG \times FI$, that is, $AX \times \overline{AX - 2AB} = 2FI \times \overline{X\gamma - AG}$. But because the Rectangles are Equal, the Sides are Proportional, $2FI$ is to $AX - 2AB$ as AX is to $X\gamma - AG$, that is, (by the third Lemma) as $X\gamma$ is to $2BC$. Q. E. D.

At length, by the Help of these Lemma's, the Construction of the Problem is thus demonstrated.

By the fourth Lemma, $X\gamma$ is to $2BC$ as $2FI$ is to $AX - 2AB$, that is, (by *Prop. 1. Book 6. Elem.*) as $2BC \times 2FI$ is to $2BC \times \overline{AX - 2AB}$, or to $2BC \times AX - 2BC \times 2AB$. But by the third Lemma, AX is to $X\gamma - 2AF$ as $X\gamma$ is to $2BC$, or $2BC \times AX = X\gamma q - 2AF \times X\gamma$, and so $X\gamma$ is to $2BC$ as $2BC \times 2FI$ is to $X\gamma q - 2AF \times X\gamma - 2BC \times 2AB$. And by multiplying the Means and Extreams into themselves, $X\gamma\, cub. - 2AF \times X\gamma q - 4BC \times AB \times X\gamma = 8BCq \times FI$. And by adding $2AF \times X\gamma q + 4BC \times AB \times X\gamma$ to both Sides $X\gamma\, cub. = 2AF \times X\gamma q + 4BC \times AB \times X\gamma + 8BCq \times FI$. But $\frac{1}{2} X\gamma$ in the

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Construction to be demonstrated was equal the Root of the Æquation $= x$, and $AF = p$, $BC = n$, $AB = \frac{q}{n}$, and $FI = \frac{r}{nn}$, and so $BC \times AB = q$. And $BCq \times FI = r$. Which being substituted, will make $x^3 = px^2 + qx + r$. Q. E. D.

Corol. Hence if AF and AB be suppos'd equal to nothing by the third and fourth Lemma, $2FI$ will be to AX as AX is to $X\gamma$, and $X\gamma$ to $2BC$. From whence arises the Invention of two mean Proportionals between any two given Quantities, FI and BC.

Scholium. Hitherto I have only expounded the Construction of a Cubick Æquation by the Ellipse; but the Rule is of a more universal Nature, extending it self indifferently to all the Conick Sections. For, if instead of the Ellipse you would use the Hyperbola, take the Lines BC and BE on the contrary Side of the Point B, then let the Points A, F, G, I, H, K, L, and R be determin'd as before, except only that FH ought to be taken on the Side of F not towards I, and that HR ought to be taken in the Line AI not in HL, on each Side the Point H, and instead of the right Line GRS, two other right Lines are to be drawn from the Point L to the two Points R and R for Asymptotes to the Hyperbola. With these Asymptotes LR, LR describe an Hyperbola through the Point G, and a Circle from the Center K with the Distance GK: And the halves of the Perpendiculars let fall from their Intersections to the right Line AE will be the Roots of the Æquation propos'd. All which, the Signs $+$ and $-$ being rightly chang'd, are demonstrated as above.

But if you would use the Parabola, the Point E will be remov'd to an infinite Distance, and so not to be taken any where, and the Point H will coincide with the Point F, and the Parabola will be to be describ'd about the Axis HL with the principal *Latus Rectum* BC through the Points G and A, the Vertex being plac'd on the same Side of the Point F, on which the Point B is in respect of the Point C.

Thus the Constructions by the Parabola, if you regard Analytick Simplicity, are the most simple of all. Those by the Hyperbola next, and those which are solv'd by the Ellipse,

lipse have the third Place. But if in describing of Figures, the Simplicity of the manual Operation be respected, the Order must be chang'd.

But it is to be observ'd in these Constructions, that by the Proportion of the principal *Latus Rectum* to the *Latus Transversum*, the Species of the Ellipse and Hyperbola may be determin'd, and that Proportion is the same as that of the Lines BC and BE, and therefore may be assum'd: But there is but one Species of the Parabola, which is obtain'd by putting BE infinitely long. So therefore we may construct any Cubick Æquation by a Conick Section of any given Species. To change Figures given in Specie into Figures given in Magnitude, is done by encreasing or diminishing all the Lines in a given Ratio, by which the Figures were given in Specie, and so we may construct all Cubick Æquations by any given Conick Section whatever. Which is more fully explain'd thus.

Let there be propos'd any Cubick Æquation $x^3 = pxx . qx . r$, to construct it by the Help of any given Conick Section. [*Vide Figures* 110 *and* 111.]

From any Point B in any infinite right Line BCE, take any two Lengths BC, and BE towards the same Way, if the Conick Section is an Ellipse, but towards contrary Ways if it be an Hyperbola. But let BC be to BE as the principal *Latus-Rectum* of the given Section, is to the *Latus Transversum*, and call BC, n, take $BA = \frac{q}{n}$, and that towards C, if q be Negative, and contrarily if Affirmative. At the Point A erect a Perpendicular AI, and in it take $AF = p$, and $FG = AF$; and $FI = \frac{r}{nn}$. But let FI be taken towards G if the Terms p and r have the same Signs, if not, towards A. Then make as FH is to FI so is BC to BE, and take this FH from the Point F towards I, if the Section is an Ellipse, but towards the contrary Way if it is an Hyperbola. But let the Parallelograms $IACK$ and $HAEL$ be compleated, and all these Lines already describ'd transferr'd to the given Conick Section; or, which is the same Thing, let the Curve be describ'd about them, so that its Axis or principal transverse Diameter might agree with the right Line LA, and the Center with the Point L. These Things being done, let the Lines KL and GL be drawn, cutting

cutting the Conick Section in g. In LK take Lk, which let be to LK as Lg to LG. and from the Center k, with the Distance kg, describe a Circle. From the Points where it cuts the given Curve, let fall Perpendiculars to the Line LH, whereof let $T\gamma$ be one. Lastly, towards γ take TY, which let be to $T\gamma$ as LG to Lg, and this TY produc'd will cut AB in X, and XY will be one of the Roots of the Æquation. But those Roots are Affirmative which lie towards such Parts of AB as FI lies from F, and those are Negative which lie on the contrary Side, if r is $+$; and the contrary if r is $-$.

After this Manner are Cubick Æquations constructed by given Ellipses and Hyperbola's: But if a Parabola should be given, the Line BC is to be taken equal to the Latus Rectum it self. Then the Points A, F, G, I, and K, being found as above, a Circle must be describ'd from the Center K with the Distance KG, and the Parabola must be so apply'd to the Scheme already describ'd, (or the Scheme to the Parabola) that it may pass through the Points A and G; and its Axis through the Point F parallel to AC, the Vertex falling on the same Side of the Point F as the Point B falls off the Point C; these being done, if Perpendiculars were let fall from the Points where the Parabola intersects the Circle to the Line BC, their Halves will be equal to the Roots of the Æquation to be constructed.

And take Notice, that where the second Term of the Æquation is wanting, and so the Latus Rectum of the Parabola is the Number 2, the Construction comes out the same as that which *Des Cartes* prov'd in his Geometry, with this Difference only, that these Lines are the double of them.

This is a general Rule of Constructions. But where particular Problems are propos'd, we ought to consult the most simple Forms of Constructions. For the Quantity n remains free, by the taking of which the Æquation may, for the most part, be render'd more simple. One Example of which I will give.

Let there be given an Ellipse, and let there be two mean Proportionals to be found between the given Lines a and b. Let the first of them be x, and $a . x . \frac{xx}{a} . b$ will be continual Proportionals, and so $ab = \frac{x^3}{a}$, or $x^3 = aab$, is the Æquation which you must construct. Here the Terms p and q are

 wanting,

wanting, and the Term $r = aab$, and therefore BA and AF are $= 0$, and $FI = \frac{aab}{nn}$. That the last Term may be more simple, let n be assum'd $= a$, and let $FI = b$. And then the Construction will be thus:

From any Point A in any infinite right Line AE, take $AC = a$, and on the same Side of the Point A take AE to AC, as the principal Latus Rectum of the Ellipse is to the Latus Transversum. Then in the Perpendicular AI take $AI = b$, and AH to AI as AC to AE. [*Vide Figure* 112.] Let the Parallelograms $IACK$, $HAEL$ be compleated. Join LA and LK. Upon this Scheme lay the given Ellipse, and it will cut the right Line AL in the Point g. Make Lk to LK as Lg to LA. From the Center k, with the Distance kg, describe a Circle cutting the Ellipse in γ. Upon AE let fall the Perpendicular γX, cutting HL in T, and let that be produc'd to Υ, that $T\Upsilon$ may be to $T\gamma$ as TA to Tg. And so $X\Upsilon = x$ will be equal to the first of the two mean Proportionals. Q. E. I.

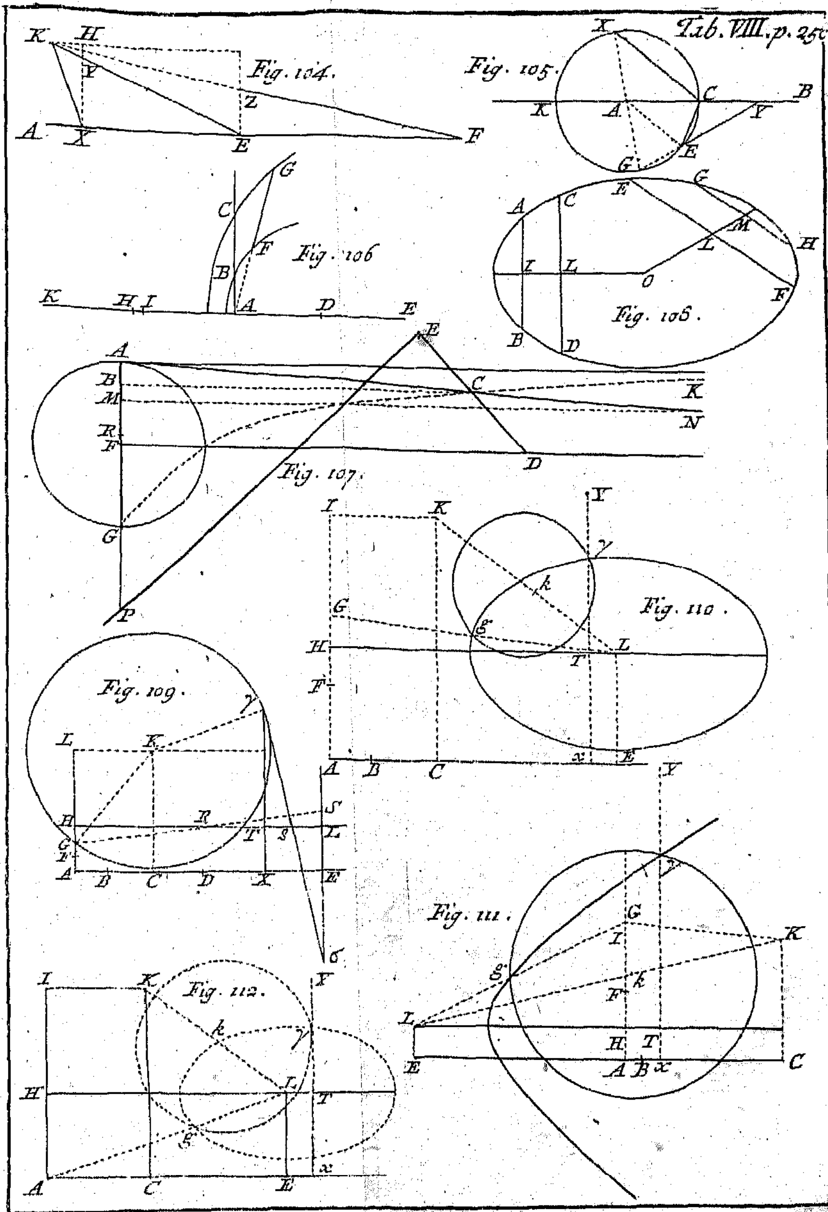
Fig. 104.
Fig. 105.
Fig. 106
Fig. 107.
Fig. 108.
Fig. 109.
Fig. 110.
Fig. 111.
Fig. 112.

A New, Exact, and Easy Method, of finding the Roots of any Æquations Generally, and that without any previous Reduction. By Edm. Halley, *Savilian Professor of Geometry.* [*Publish'd in the* Philosophical Transactions, Numb. 210. A. D. 1694.]

THE principal Use of the Analytick Art, is to bring Mathematical Problems to Æquations, and to exhibit those Æquations in the most simple Terms that can be. But this Art would justly seem in some Degree defective, and not sufficiently Analytical, if there were not some Methods, by the Help of which, the Roots (be they Lines or Numbers) might be gotten from the Æquations that are found, and so the Problems in that respect be solved. The Antients scarce knew any Thing in these Matters beyond Quadratick Æquations. And what they writ of the Geometrick Construction of solid Problems, by the Help of the Parabola, Cissoid, or any other Curve, were only particular Things design'd for some particular Cases. But as to Numerical Extraction, there is every where a profound Silence; so that whatever we perform now in this Kind, is entirely owing to the Inventions of the Moderns.

And first of all, that great Discoverer and Restorer of the Modern Algebra, *Francis Vieta*, about 100 Years since, shew'd a general Method for extracting the Roots of any Æquation, which he publish'd under the Title of, *A Numerical Resolution of Powers, &c. Harriot, Oughtred*, and others, as well of our own Country, as Foreigners, ought to acknowledge whatsoever they have written upon this Subject, as taken from *Vieta*. But what the Sagacity of Mr. *Newton's* Genius has perform'd in this Business we may rather conjecture (than be fully assur'd of) from that short Specimen

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given by Dr. *Wallis* in the 94th Chapter of his *Algebra*. And we must be forc'd to expect it, till his great Modesty shall yield to the Intreaties of his Friends, and suffer those curious Discoveries to see the Light.

Not long since, (*viz. A. D.* 1690,) that excellent Person, Mr. *Joseph Ralphson*, F. R. S. publish'd his *Universal Analysis of Æquations*, and illustrated his Method by Plenty of Examples; by all which he has given Indications of a Mathematical Genius, from which the greatest Things may be expected.

By his Example, M. *de Lagney*, an ingenious Professor of Mathematicks at *Paris*, was encourag'd to attempt the same Argument; but he being almost altogether taken up in extracting the Roots of pure Powers (especially the Cubick) adds but *little* about affected Æquations, and that pretty much perplex'd too, and not sufficiently demonstrated: Yet he gives two very compendious Rules for the Approximation of a Cubical Root; one a Rational, and the other an Irrational one. *Ex. gr.* That the Side of the Cube $aaa + b$ is between $a + \frac{ab}{3aaa + b}$, and $\sqrt{\frac{1}{4}aa + \frac{b}{3a}} + \frac{1}{2}a$. And the Root of the 5th Power, $a^5 + b$, he makes $= \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$ (where note, that 'tis $\frac{1}{4}aa$, not $\frac{1}{2}aa$, as 'tis erroneously printed in the French Book.) These Rules were communicated to me by a Friend, I having not seen the Book; but having by Trial found the Goodness of them, and admiring the Compendium, I was willing to find out the Demonstration. Which having done, I presently found that the same Method might be accommodated to the Resolution of all Sorts of Æquations. And I was the rather inclin'd to improve these Rules, because I saw that the whole Thing might be explain'd in a *Synopsis*; and that by this means, at every repeated Step of the Calculus, the Figures already found in the Root, would be at least trebled, which all other Ways are encreased but in an equal Number with the given ones. Now, the 'forementio n'd Rules are easily demonstrated from the Genesis of the Cube, and the 5th Power. For, supposing the Side of any Cube $= a + e$, the Cube arising from thence is $aaa + 3aae + 3aee + eee$. And consequently, if we suppose aaa the next less Cube, to any given Non-Cubick Number, then eee will be less than

Unity,

Unity, and the Remainder b, will $=$ the other Members of the Cube, $3aae+3aee+eee$. Whence rejecting eee upon the Account of its Smallness, we have $b=3aae+3aee$. And since aae is much greater than aee, the Quantity $\frac{b}{3aa}$ will not much exceed e; so that putting $e=\frac{b}{3aa}$ then the Quantity $\frac{b}{3aa+3ae}$ (to which e is nearly equal) will be found $=\frac{b}{3aa+\frac{3ab}{3aa}}$, or $\frac{b}{3aa+\frac{b}{a}}$, that is, $\frac{ab}{3aa+b}=e$. And so the Side of the Cube $aaa+s$ will be $a+\frac{ab}{3aa+b}$, which is the *Rational Formula* of M. *de Lagney.* But now, if aaa were the next greater Cubick Number to that given, the Side of the Cube $aaa-b$, will, after the same Manner, be found to be $a-\frac{ab}{3aa-b}$: And this easy and expeditious Approximation to the Cubick Root, is only (a very small Matter) erroneous in point of *Defect*, the Quantity e, the Remainder of the Root thus found, coming something less than really it is.

As for the *Irrational Formula*, 'tis deriv'd from the same Principle, *viz.* $b=3aae+3aee$, or $\frac{b}{3a}=ae+ee$, and so $\sqrt{\frac{1}{4}aa+\frac{b}{3a}}=\frac{1}{2}a+e$, and $\sqrt{\frac{1}{4}aa+\frac{b}{3a}}+\frac{1}{2}a=a+e$, the Root sought. Also the Side of the Cube $aaa-b$, after the same Manner, will be found to be $\frac{1}{2}a+\sqrt{\frac{1}{4}aa-\frac{b}{3a}}$. And this *Formula* comes something nearer to the Scope, being erroneous in point of *Excess*, as the other was in *Defect*, and is more accommodated to the Ends of Practice, since the Restitution of the Calculus is nothing else but the continual Addition or Subtraction of the Quantity $\frac{aee}{3a}$, according as the Quantity e can be known. So that we should rather write $\sqrt{\frac{1}{4}aa + \frac{b - eee}{3a}} + \frac{1}{2}a$, in the former Case, and in the latter, $\frac{1}{2}a + \sqrt{\frac{1}{4}aa + \frac{eee - b}{3a}}$. But by either of the two *Formula's* the Figures already known in the Root to be extracted are at least tripled; which I conclude will be very grateful to all the Students in Arithmetick, and I congratulate the Inventor upon the Account of his Discovery.

But that the Use of these Rules may be the better perceiv'd, I think it proper to subjoyn an Example or two. Let it be propos'd to find the Side of the double Cube, or $aaa + b = 2$. Here $a = 1$, and $\frac{b}{3a} = \frac{1}{3}$, and so $\frac{1}{2} + \sqrt{\frac{7}{12}}$, or $1{,}26$, be found to be the true Side nearly. Now, the Cube of $1{,}26$, is $2{,}000376$, and so $0{,}63 + \sqrt{,3969 - \frac{,000376}{3{,}78}}$ or $0{,}63 + \sqrt{,\dot{3}96\dot{8}00\dot{5}2\dot{9}1\dot{0}0\dot{5}291} = 1{,}259921049895 -$; which in 13 Figures gives the Side of the double Cube with very little Trouble, *viz.* by one only Division, and the Extraction of the Square Root; when as by the common Way of working, how much Pains it would have cost, the Skillful very well know. This Calculus a Man may continue as far as he pleases, by encreasing the Square by the Addition of the Quantity $\frac{eee}{3a}$; which Correction, in this Case, will give but the Encrease of Unity in the 14th Figure of the Root.

Example II. Let it be propos'd to find the Sides of a Cube equal to that English Measure commonly call'd a Gallon, which contains 231 solid Ounces. The next less Cube is 216, whose Side $6 = a$, and the Remainder $15 = b$; and so for the first Approximation, we have $3 + \sqrt{9 + \frac{5}{6}} =$ the Root. And since $\sqrt{9{,}8333\ldots}$ is $3{,}1358\ldots$, 'tis plain, that $6{,}358 = a + e$. Now, let $6{,}1358 = a$; and we shall then have for its Cube $231{,}000853894712$, and according to the Rule, $3{,}0679 + \sqrt{9{,}41201041 - \frac{,000858394712}{18{,}4070}}$ is most accurately equal to the Side of the given Cube, which, within the Space of an Hour, I determin'd by Calculation to be be 0.13579243966195897, which is exact in the 18th Figure, defective in the 19th. And this *Formula* is deservedly preferable to the *Rationale*, upon the Account of the great Divisor, which is not to be manag'd without a great deal of Labour; whereas the Extraction of the Square Root proceeds much more easily, as manifold Experience has taught me.

But the Rule for the Root of a pure Sursolid, or the 5th Power, is of something a higher Enquiry, and does much more perfectly yet do the Business; for it does at least Quintuple the given Figures of the Root, neither is the Calculus very large or operose. Tho' the Author no where shews his Method of Invention, or any Demonstration, altho' it seems to be very much wanting; especially since all Things are not right in the printed Book, which may easily deceive the Unskilful. Now the 5th Power of the Side $a+e$ is compos'd of these Members, $a^5 + 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4 + e^5 = a^5 + b$; from whence $b = 5a^4e + 10a^3e^2 + 10a^2e^3 + 5ae^4$, rejecting e^5 because of its Smallness. Whence $\frac{b}{5a} = a^3e + 2a^2e^2 + 2ae^3 + e^4$, and adding on both Sides $\frac{1}{4}a^4$, we shall have $\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} = \sqrt{\frac{1}{4}a^4 + a^3e + {}^2a^2e^2 + 2ae^3 + e^4} = \frac{1}{2}aa + ae + ee$. Then subtracting $\frac{1}{4}aa$ from both Sides, $\frac{1}{2}a + e$ will $= \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$; to which, if $\frac{1}{2}a$ be added, then will $a + e = \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa} =$ the Root of the Power $a^5 + b$. But if it had $a^5 - b$ (the Quantity a being too great) the Rule would have been thus, $\frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 - \frac{b}{5a}} - \frac{1}{4}aa}$. And this Rule approaches wonderfully, so that there is hardly any need of Restitution.

But while I consider'd these Things with my self, I light upon a general Method for the *Formula's* of all Powers whatsoever, and (which being handsome and concise enough) I thought I would not conceal from the Publick.

These

These Formula's, (as well the *Rational* as the *Irrational* ones) are thus.

$$\sqrt{aa+b} = \sqrt{aa+b}, \text{ or } a + \frac{ab}{2aa + \frac{1}{2}b}.$$

$$\sqrt[3]{a^3+b} = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + \frac{b}{3a}}, \text{ or } a + \frac{ab}{3aaa + b}.$$

$$\sqrt[4]{a^4+b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa + \frac{b}{6aa}}, \text{ or } a + \frac{ab}{4a^4 + \frac{3}{2}b}.$$

$$\sqrt[5]{a^5+b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa + \frac{b}{10a^3}}, \text{ or } a + \frac{ab}{5a^5 + 2b}.$$

$$\sqrt[6]{a^6+b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa + \frac{b}{15a^4}}, \text{ or } a + \frac{ab}{6a^6 + \frac{5}{2}b}.$$

$$\sqrt[7]{a^7+b} = \tfrac{5}{6}a + \sqrt{\tfrac{1}{36}aa + \frac{b}{21a^5}}, \text{ or } a + \frac{ab}{7a^7 + 3b}.$$

And so also of the other higher Powers. But if a were assum'd bigger than the Root sought, (which is done with some Advantage, as often as the Power to be resolv'd is much nearer, the Power of the *next greater* whole Number, than of the *next less*) in this Case, *Mutatis Mutandis*, we shall have the same Expressions of the Roots, *viz.*

$$\sqrt{aa-b} = \sqrt{aa-b}, \text{ or } a - \frac{ab}{2aa - \frac{1}{2}b}.$$

$$\sqrt[3]{a^3-b} = \tfrac{1}{2} + \sqrt{\tfrac{1}{4}aa - \frac{b}{3a}}, \text{ or } a - \frac{ab}{3a^3 - b}.$$

$$\sqrt[4]{a^4-b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa - \frac{b}{6aa}}, \text{ or } a - \frac{ab}{4a^4 - \frac{3}{2}b}.$$

$$\sqrt[5]{a^5-b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa - \frac{b}{10a^3}}, \text{ or } a - \frac{ab}{5a^5 - 2b}.$$

$$\sqrt[6]{a^6-b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa - \frac{b}{15a^4}}, \text{ or } a - \frac{ab}{6a^6 - \frac{5}{2}b}.$$

$$\sqrt[7]{a^7-b} = \tfrac{5}{6}a + \sqrt{\tfrac{1}{36}aa - \frac{b}{21a^5}}, \text{ or } a - \frac{ab}{7a^7 - 3b}.$$

And

And within these two Terms the true Root is ever found, being something nearer to the *Irrational* than the *Rational* Expression. But the Quantity e found by the *Irrational Formula*, is always too great, as the Quotient resulting from the *Rational Formula*, is always too little. And consequently, if we have $+b$, the *Irrational Formula* gives the Root something greater than it should be, and the *Rational* something less. But contrarywise if it be $-b$.

And thus much may suffice to be said concerning the Extraction of the Roots of pure Powers; which notwithstanding, for common Uses, may be had much more easily by the Help of the Logarithms. But when a Root is to be determin'd very accurately, and the Logarithmick Tables will not reach so far, then we must necessarily have Recourse to these, or such like Methods. Farther, the Invention and Contemplation of these *Formula's* leading me to a certain universal Rule for adfected Æquations, (which I hope will be of Use to all the Students in *Algebra* and *Geometry*) I was willing here to give some Account of this Discovery, which I will do with all the Perspicuity I can. I had given at N° 188. of the *Transactions*, a very easy and general Construction of all adfected Æquations, not exceeding the Biquadratick Power; from which Time I had a very great Desire of doing the same in Numbers. But quickly after, Mr. *Ralphson* seem'd in great Measure to have satisfy'd this Desire, till Mr. *Lagney*, by what he had perform'd in his Book, intimated, that the Thing might be done more compendiously yet. Now, my Method is thus:

Let z, the Root of any Æquation, be imagin'd to be compos'd of the Parts $a+$, or $-e$, of which, let a be assum'd as near z as is possible; which is notwithstanding not *necessary*, but only *commodious*. Then from the Quantity $a+e$, or $a-e$, let there be form'd all the Powers of z, found in the Æquation, and the Numerical Co-efficients be respectively affix'd to them: Then let the Power to be *resolv'd* be subtracted from the Sum of the given Parts (in the first Column where e is not found) which they call the *Homogeneum Comparationis*, and let the Difference be $\pm b$. In the next Place, take the Sum of all the Co-efficients of e in the second Column, to which put $=s$. Lastly, in the third Column let there be put down the Sum of all the Co-efficients of ee, which Sum call t. Then will the Root z stand thus in the *Rational Formula*, viz. $z=a+\frac{sb}{ss\pm tb}$; and

 thus

thus in the *Irrational Formula*, viz. $z = a \mp \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \mp bt}}{t}$; which perhaps it may be worth while to illustrate by some Examples. And instead of an *Instrument* let this *Table* serve, which shews the Genesis of the several Powers of $a \pm e$, and if need be, may easily be continued farther; which, for its Use, I may rightly call a *General Analytical Speculum.* The forementioned Powers arising from a continual Multiplication by $a + e$ ($= z$) come out thus with their adjoyn'd Co-efficients.

Tabula Potestatum.

		s	t	u	w	x	y	
$lz^7 =$	la^7	$+ 7la^6e$	$+ 21la^5ee$	$+ 35la^4e^3$	$+ 35la^3e^4$	$+ 21la^2e^5$	$+ 7lae^6$	$+ le^7$
$kz^6 =$	ka^6	$+ 6ka^5e$	$+ 15ka^4ee$	$+ 20ka^3e^3$	$+ 15ka^2e^4$	$+ 6kae^5$	$+ ke^6$	
$hz^5 =$	ha^5	$+ 5ha^4e$	$+ 10ha^3ee$	$+ 10ha^2e^3$	$+ 5hae^4$	$+ he^5$		
$gz^4 =$	ga^4	$+ 4ga^3e$	$+ 6ga^2ee$	$+ 4gae^3$	$+ ge^4$			
$fz^3 =$	fa^3	$+ 3fa^2e$	$+ 3faee$	$+ fe^3$				
$dz^2 =$	da^2	$+ 2dae$	$+ dee$					
$cz =$	ca	$+ ce$						

But now, if it be $a-e=z$, the Table is compos'd of the same Members, only the odd Powers of e, as e, e^3, e^5, e^7 are Negative, and the even Powers, as e^2, e^4, e^6, Affirmative. Also, let the Sum of the Co-efficients of the Side e, be $=s$; the Sum of the Co-efficients of the Square ee $=t$, the Sum of the Co-efficient of $e^3=u$, of $e^4=w$, of $e^5=x$, of $e^6=y$, &c. But now, since e is suppos'd only a small Part of the Root that is to be enquir'd, all the Powers of e will be much less than the correspondent Powers of a, and so far the first Hypothesis; all the superior ones may be rejected; and forming a new Æquation, by substituting $a\pm e=z$, we shall have (as was said) $\pm b=\pm se\pm tee$. The following Examples will make this more clear.

EXAMPLE I. Let the Æquation $z^4-3z^2+75z=10000$ be propos'd. For the first Hypothesis, let $a=10$, and so we have this Æquation;

$$\begin{array}{llllll}
z^4 & =+a^4 & 4a^3e & +6a^2ee & 4ae^3 & +e^4\\
-dz^2 & =-da^2 & da\,e & -d\,ee & & \\
+cz & =+ca & c\,e & & & \\
 & =+10000 & 4000e & +600ee & 40e^3 & +e^4\\
 & -300 & 60e & -3ee & & \\
 & +750 & 75e & & & \\
 & -10000 & & & &
\end{array}$$

$$+\ 450-\underset{s}{4015}e+\underset{t}{597}ee-\underset{u}{40}e^3+e^4=0$$

The Signs $+$ and $-$, with respect to the Quantities e and e^3, are left as doubtful, till it be known whether e be Negative or Affirmative; which Thing creates some Difficulty, since that in Æquations that have several Roots, the *Homogenea Comparationis* (as they term them) are oftentimes encreased by the minute Quantity a, and on the contrary, *that* being encreased, *they* are diminish'd. But the Sign of e is determin'd from the Sign of the Quantity b. For taking away the *Resolvend* from the *Homogeneal* form'd of a; the Sign of se (and consequently of the prevailing Parts in the Composition of it) will always be contrary to the Sign of the Difference b. Whence 'twill be plain, whether it must be $+e$, or $-e$; and consequently, whether a be taken greater or less than the *true Root*. Now the Quantity e is $=\frac{1}{2}s\mp\sqrt{\dfrac{\frac{1}{4}ss-bt}{t}}$, when b and t have the same Sign, but

 when

when the Signs are different, e is $= \sqrt{\frac{\frac{1}{4}ss + bt}{t}} - \frac{1}{2}s$. But after it is found that it will be $-e$, let the Powers e, e^3, e^5, &c. in the affirmative Members of the Æquation be made Negative, and in the Negative be made Affirmative; that is, let them be written with the contrary Sign. On the other hand, if it be $+e$ (let those forementioned Powers) be made Affirmative in the Affirmative, and Negative in the Negative Members of the Æquation.

Now we have in this Example of ours, 10450 instead of the Resolvend 10000, or $b = +450$, whence it's plain, that a is taken greater than the Truth, and consequently, that 'tis $-e$. Hence the Æquation comes to be, $10450 - 4015e + 597ee - 4e^3 + e^4 = 10000$. That is, $450 - 4015e + 597ee = 0$; and so $450 = 4015e - 597ee$, or $b = se - tee$, whose Root $e = \frac{1}{2}s - \sqrt{\frac{\frac{1}{4}ss - bt}{t}}$, or $\frac{s}{2t} - \sqrt{\frac{ss}{4tt} - \frac{b}{t}}$; that is, in the present Case,

$e = \frac{2007\frac{1}{2} - \sqrt{3761406\frac{1}{4}}}{597}$, from whence we have the Root sought, 9,886, which is near the Truth. But then substituting this for a second Supposition, there comes $a + e = z$, most accurately, 9,8862603936495 .... scarce exceeding the Truth by 2 in the last Figure, *viz.* when $\sqrt{\frac{\frac{1}{4}ss + bt}{t}} - \frac{1}{2}s = e$. And this (if need be) may be yet much farther verify'd, by subtracting (if it be $+e$) the Quantity $\frac{\frac{1}{2}ue^3 + \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss + tb}}$, from the Root before found; or (if it be $-e$) by adding $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - tb}}$ to that Root. Which Compendium is so much the more valuable, in that sometimes from the first Supposition alone, but always from the second, a Man may continue the Calculus (keeping the same Co-efficients) as far as he pleases. It may be noted, that the fore-mention'd Æquation has also a Negative Root, *viz.* $z = 10,26$ .... which any one that has a Mind, may determine more accurately.

EXAMPLE

EXAMPLE II. Suppose $z^3 - 17z + 54z = 350$, and let $a = 10$. Then according to the Prescript of the Rule,

$$\begin{aligned} +z^3 &= a^3 + 3a^2e + 3aee + e^3 \\ -dz^2 &= da^2 - 2dae - de^2 \\ +cz &= ca + ce \end{aligned}$$

$$\begin{array}{lllll} & b & s & t & \\ \text{That is,} & +1000 & +300e & +30e^2 & +e^3 \\ & -1700 & -340e & -17e^2 & \\ & +540 & +54e & & \\ & -350 & & & \\ \hline \text{Or,} & -510 & +14e & +13ee & +e^3 = 0 \end{array}$$

Now, since we have -510, it is plain, that a is assumed less than the Truth, and consequently that e is Affirmative. And from (the Æquation) $510 = 14e + 13e^2$, comes $e = \frac{\sqrt{bt + \frac{1}{4}ss} - \frac{1}{2}s}{t} = \frac{\sqrt{6679} - 7}{13}$. Whence $z = 15,7 \ldots$, which is too much, because of a taken wide. Therefore, Secondly, let $a = 15$, and by the like Way of Reasoning we shall find $e = \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - tb}}{t} = \frac{109\frac{1}{2} - \sqrt{11710\frac{1}{4}}}{28}$, and consequently, $z = 14,954068$. If the Operation were to be repeated the third Time, the Root will be found conformable to the Truth as far as the 25th Figure; but he that is contented with fewer, by writing $tb \pm te^3$ instead of tb, or subtracting or adding $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss \mp tb}}$ to the Root before found, will presently obtain his End. Note, the Æquation propos'd is not explicable by any other Root, because the *Resolvend* 350 is greater than the Cube of $\frac{17}{3}$, or $\frac{d}{3}$.

EXAMPLE III. Let us take the Æquation $z^4 - 80z^3 + 1998z^2 - 14937z + 5000 = 0$, which Dr. *Wallis* uses *Chap.* 62. of his *Algebra*, in the Resolution of a very difficult Arithmetical Problem, where, by *Vieta*'s Method, he has obtain'd the Root most accurately; and Mr. *Ralphson* brings it also as an Example of his Method, *Page* 25, 26. Now this Æquation is of the Form which may have several Affirmative Roots, and (which increases the Difficulty) the *Co-efficients* are very great in respect of the *Resolvend* given.

But

But that it may be the easier manag'd, let it be divided, and according to the known Rules of *Pointing*, let $-z^4 + 8z^3 - 20z^2 + 15z = 0,5$ (where the Quantity z is $\frac{1}{10}$ of z in the Æquation propos'd) and for the first Supposition, let $a = 1$. Then $+2 - 5e - 2e^2 + 4e^3 - e^4 - 0,5 = 0$; that is, $1\frac{1}{2} = 5e + 2ee$; hence $e = \frac{\sqrt{\frac{1}{4}ss + bt} - \frac{1}{2}s}{t}$ is $= \frac{\sqrt{37} - 5}{4}$, and so $z = 1,27$; whence 'tis manifest, that 12,7 is near the true Root of the Æquation propos'd. Now, Secondly, let us suppose $z = 12,7$, and then according to the Directions of the Table of Powers, there arises

$$\begin{array}{llll}
\quad b & \quad s & \quad t & \quad u \\
-26014,4641 & -8193,532e & -967,74e^2 & -50,8e^3 - e^4 \\
+153870,640 & +38709,60\,e & +3048\,e^2 & +80\,e^3 \\
-322257,42 & -50749,2\,e & -1998\,e^2 & \\
+189699,9 & +14937,\,e & & \\
-5000. & & & \\
\hline
+298,6559 & -5296,132e & +82,26e^2 & +29,2e^3 - e^4 = 0.
\end{array}$$

And so $-298,6559 = -5296,132e + 82,26ee$, whose Root e (according to the Rule) $= \frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt}}{t}$, comes to $\frac{2648,066 - \sqrt{6987686,106022}}{82,26} = ,0564408033 \ldots$ $= e$ less than the Truth. But that it may be corrected, 'tis to be consider'd, that $\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - bt}}$, or $\frac{,0026201 \ldots\ldots}{2643,423 \ldots\ldots}$ is ,00000099, and consequently e corrected, is $= 0564470448$. And if you desire yet more Figures of the Root, from the e corrected, let there be made $tue^3 - te^4 = 0,4310560242 3 \ldots$, and $\frac{\frac{1}{2}s - \sqrt{\frac{1}{4}ss - bt - tue^3 + te^4}}{t}$, or which is all one,

$$\frac{2648,066 - \sqrt{6987685,67496597577 \ldots}}{82,26} =$$

$,05644179448074402 = e$; whence $a + e = z$ the Root is most accurately $12,75644179448074402 \ldots$, as Dr. *Wallis* found in the forementiond Place; where it may be observ'd, that the Repetition of the *Calculus* does ever triple the true Figures in the assum'd a, which the first Correction, or $\frac{1}{2}ue$

$\frac{\frac{1}{2}ue^3 - \frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss - bt}}$ does quintuple; which is also commodiously done by the *Logarithms*. But the other Correction after the first, does also double the Number of Figures, so that it renders the *assumed* altogether Seven-fold; yet the first Correction is abundantly sufficient for Arithmetical Uses, for the most Part.

But as to what is said concerning the Number of Places rightly taken in the Root, I would have understood so, that when a is but $\frac{1}{10}$ Part distant from the true Root, then the first Figure is rightly assumed; if it be within $\frac{1}{100}$ Part, then the two first Figures are rightly assumed; if within $\frac{1}{1000}$, and then the three first are so; which consequently, manag'd according to our Rule, do presently become nine Figures.

It remains now that I add something concerning our *Rational Formula*, viz. $e = \frac{sb}{ss \pm tb}$, which seems expeditious enough, and is not much inferior to the former, since it will triple the given Number of Places. Now, having form'd an Æquation from $a \pm e = z$, as before, it will presently appear, whether a be taken greater or lesser than the Truth; since se ought always to have a Sign contrary to the Sign of the Difference of the *Resolvend*, and its *Homogeneal* produc'd from a. Then supposing $+b + se + a - tee = 0$, the Divisor is $ss - tb$, as often as t and b have the same Signs; but it is $ss + bt$, when they have different ones. But it seems most commodious for Practice, to write the *Theorem* thus, $e = \frac{b}{s} + \frac{tb}{s}$, since this Way the Thing is done by one Multiplication and two Divisions, which otherwise would require three Multiplications, and one Division.

Let us take now one Example of this Method, from the Root (of the forementioned Æquation) 12,7 ...., where

$$\begin{array}{lllll} 298,6559 & - 5296,132e & + 82,26ee & + 29,2e^3 - e^4 = 0, \\ +b & -s & +t & +u \end{array}$$

and so $\frac{b}{s} - \frac{tb}{s} = e$; that is, let it be as s to t, so b to $\frac{tb}{s} = 5296,132$) 298,6559 into 82,26 (4,63875 ... wherefore the Divisor is $s - \frac{tb}{s} = 5291,49325 \ldots\ldots$) 298,6559

(0,056441

(0,056441 ......) = e, that is, to five true Figures, added to the Root that was taken. But this *Formula* cannot be corrected, as the foregoing *Irrational* one was; and so if more Figures of the Root are desired, 'tis the best to make a new Supposition, and repeat the *Calculus* again: And then a new Quotient, tripling the known Figures of the Root, will abundantly satisfy even the most Scrupulous.

FINIS.
