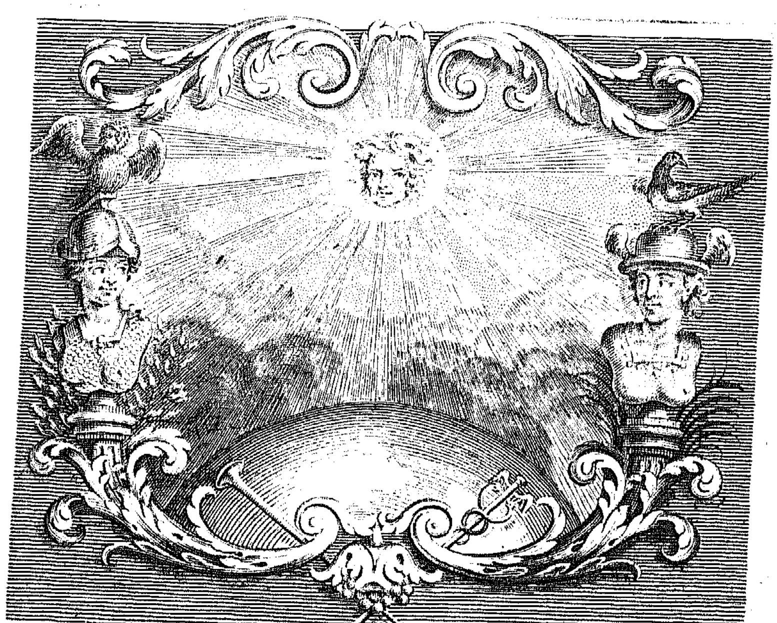
Newton
Craig
Cotes

ANALYSIS
Per Quantitatum
SERIES, FLUXIONES,
AC
DIFFERENTIAS:
CUM
Enumeratione Linearum
TERTII ORDINIS.



J. Nutting Sculp.

LONDINI:
Ex Officina PEARSONIANA. *Anno* M.DCC.XI.

Præfatio Editoris.

Tractatus aliquot Mathematicos Viri Illustrissimi D. Newtoni *in lucem edimus, quorum primus & ultimus nunc primum prodeunt, reliqui vero vel à se vel ab aliis ante hac publici juris facti sunt. Hæc autem Editio casui, sed tamen non sine ipsius consensu prius impetrato, ortum acceptum refert.*

Etenim secundus jam agitur annus ex quo scrinia D. Collinsii (*qui, uti notum est, amplissimum cum sui sæculi Mathematicis commercium habuit*) *meas in manus inciderunt; & in illis plurima reperi à cunctis fere totius* Europæ *eruditis ipsi communicata; & inter ea non pauca, quæ à Viro Cl* D. Newtono *scripta fuerunt; quæ cum tantæ molis essent, ut simul Tractatum breviusculum possent conficere, cæpi de iis* *edendis cogitare. Quum autem animadvertissem scripta ejus quæ jam in lucem prodierunt ferme idem cum hisce argumentum habere, haud operæ pretium me facturum, si typis mandarem, existimavi.*

Unus tamen erat brevis de Curvarum Quadratura Tractatus adeo luculenter & concinne scriptus, atque ita accommodatus ad instituendos eos, qui nondum totam istam Methodum perspectam habeant, ut abstinere non potuerim, quominus Auctoris licentiam eundem edendi peterem. Quam Ille non solum summa cum humanitate concessit; sed & insuper veniam dedit reliqua ipsius colligendi, quæ ad idem argumentum spectabant.

Hicce Tractatus, quem D. Collinsii *manu exaratum comperi, inscriptus fuit* De Analysi per Æquationes Infinitas, *& licet neque Auctoris nomen, neque tempus quo scriptus fuerat ullibi comparuerit; multa tamen continere ad* D. Newtoni *Methodos spectantia statim agnovi, utpote Epistolarum autographo illi ad amussim respondentia, quod antea ipse* D. Newtonus *ad D.* Oldenburgum *miserat. Unde suspicabar totum ex eodem fonte emanasse; nihilominus suspenso eram animo, usque dum inventis, inter easdem chartas, Epistolis aliquot* D. Barrovii *ad D.* Collinsium *scriptis; tres reperi ad hunc Tractatum*

ctatum immediate ſpectantes; ex quibus facile intellexi quomodo D. Collinſius *eum obtinuerat.*

In una Epiſtolarum e Collegio S. S. Tr. data 20 Julii 1669, *verſus finem* D. Barrovius *hæc ſcribit.*

* Amicus quidam apud nos commorans, ‘ qui eximio in his rebus pollet ingenio, nu‑ ‘ diuſtertius chartas quaſdam mihi tradidit, ‘ in quibus Magnitudinum dimenſiones ‘ ſupputandi Methodos, *Mercatoris* methodo ‘ ſimiles, maxime vero Generales, deſcripſit, ‘ ſimulque Æquationes reſolvendi, quæ, ut ‘ opinor, tibi placebunt, quas una cum ‘ proximis literis ad te mittam.

In hac Epiſtola narrat argumentum chartarum Amici ſui fuiſſe Computationem dimenſionum Magnitudinum, & Æquationum reſolutionem, quod cum Tractatus hujus argumento quadrat.

Verſus finem alterius Epiſtolæ ad D. Collinſium *e Collegio S. S. Tr. datæ ult.* Julii 1669, D. Barrovius *ita ſcribit.*

Mitto

* A Friend of mine here, that hath an excellent Genius to theſe things, brought me the other Day, ſome Papers, wherein he hath ſet down Methods of Calculating the Dimenſions of Magnitudes, like that of Mr. *Mercator*, but very general, as alſo of Reſolving Equations, which I ſuppoſe will pleaſe you, and I ſhall ſend you them by the next.

PRÆFATIO.

† Mitto quas pollicitus eram Amici char-'tas, quæ uti ſpero haud parum te obleċta-'bunt. Remittas, quæſo, quum eas quantum 'tibi viſum fuerit perlegeris; id enim poſtu-'lavit Amicus meus, cum primum eum ro-'gavi, ut eas tecum communicare mihi lice-'ret. Quantocyus igitur, obſecro, te eas 'recepiſſe fac me certiorem, quod illis me-'tuo, quippe qui eas per Veredarium publi-'cum ad te mittere non dubitaverim, quo 'tibi morem gererem quam citiſſime.

Ex hac Epiſtola conſtat D. Barrovium *dictum Librum miſiſſe, ea lege, ut ſibi remitteretur. Unde manifeſtum eſt, quare Tractatus, quem inveni,* D. Collinſii *manu ſcriptus fuit, autographo nempe ipſi Auctori reſtituto.*

In tertia a D. Barrovio *ad* D. Collinſium *Epiſtola data* 20 Auguſti 1669, *conſtat* D. Newtonum *fuiſſe, Amicum illum, de quo* D. Barrovius *in duabus prioribus Epiſtolis mentionem fecerat, quod his verbis conſignat.*

Amici

† I ſend you the Papers of my Friend I promis'd, which I preſume will give you much ſatisfaction; I pray, having peruſed them ſo much as you think good, remand them to me, according to his deſire when I ask'd him the Liberty to impart them to you; I pray give me notice of your receiving *them*, with your ſooneſt convenience, that I may be ſatisfied of their reception; becauſe I am afraid of them, venturing them by the Poſt, that I may not longer delay to correſpond with your deſire.

* Amici chartas tibi placuisse gaudeo;
' est illi nomen *Newtonus*, Collegii nostri So-
' cius, & juvenis, (secundus enim, ex quo
' Artium Magistri gradum cepit, jam agitur
' annus,) et qui, eximio quo est acumine,
' permagnos in hac re progressus fecit. Illas,
' si vis, cum Nobili Domino Vicecomite
' *Brounkero* communica.

Perspecto jam D. Newtonum *hujus Tractatus Auctorem esse; ab eo sciscitatus sum num penes se adhuc esset autographum, quod quidem ille exquirens invenit, & mihi tradidit, cum exemplari* Collinsiano *ad verbum usque conveniens.*

Etiamsi vero hic Tractatus ad D. Collinsium *missus fuisset mense* Julii 1669, *quod aliquantulum erat posteaquam D.* Mercator *Logarithmotechniam suam in lucem ediderat; manifestum est ex quibusdam aliis Epistolis, (quæ itidem inter D.* Collinsii *chartas fuerunt,) quod antea scriptus esset, imo quod D.* Newtonus *invenisset Methodum investigandi Magnitudinum Dimensiones per Infinitas Series vel aliquot annos antequam D.* Mercator *Librum suum in vulgus edidit; ut liquet ex Epistola a* Collinsio

b *ad*

* I am glad my Friend's Paper gives you so much satisfaction, his Name is Mr. *Newton*, a Fellow of our College, and very young, (being but the second Year Master of *Arts*,) but of an extraordinary Genius and Proficiency in these things; you may impart the Papers, if you please, to my Lord *Brounker*.

ad D. Jacobum Gregorium *data* 25. Novemb. 1669, *ubi hæc ſunt verba.*

* *Barrovius* Provinciam ſuam Publice præ-
‘legendi remiſit cuidam nomine *Newtono*
‘*Cantabrigienſi*, quem tanquam Virum acutiſſi-
‘mi ingenii, in Præfatione Prælectionum Op-
‘ticarum memorat, & qui, antequam edere-
‘tur *Mercatoris* Logarithmotechnia, Metho-
‘dum invenerat eandem, eamque ad omnes
‘Curvas generaliter, & ad Circulum diver-
‘ſimode applicârat.’

Quinetiam D. Collinſius *in Epiſtola ad D.* Strode, *data* 26. Julii 1672, *ſic ſcribit.*

† Menſe *Septembri* 1668, *Mercator* Loga-
‘rithmotechniam edidit ſuam, quæ ſpeci-
‘men hujus Methodi (i. e. Serierum Infini-
‘tarum) in unica tantum Figura, nempe,
‘Quadratura Hyperbolæ continet. Haud
‘multo poſtquam in publicum prodiret liber,
‘exemplar unum Cl. *Walliſio Oxoniam* miſi,
‘qui ſuum de eo judicium in *Actis Philo-*
‘*ſophicis* ſtatim fecit: alterum *Barrovio Can-*
‘*tabrigiam*, qui quaſdam *Newtoni* chartas, qui
‘jam *Barrovium* in Mathematicis Prælecti-
‘onibus

* Mr. *Barrow* hath reſign'd his Lecture's Place *to one* Mr. *Newton* of *Cambridge*, whom he mentions in his Optic Preface as a very ingenious Perſon; one who hath, before Mr. *Mercator*'s Logarithmotechnia was extant, invented the ſame Method, and applied it generally to all Curves, and divers ways to the Circle.

'onibus publicis excipit, extemplo remisit: 'Ex quibus & aliis, quæ olim ab Auctore 'cum *Barrovio* communicata fuerant, patet 'illam Methodum a dicto *Newtono* aliquot 'annis ante excogitatam, & modo generali 'applicatam fuisse: ita ut ejus ope in quavis 'Figura Curvilinea proposita, quæ una vel 'pluribus Proprietatibus definitur, Quadra-'tura vel Area dictæ Figuræ, accurata si 'possibile sit, sin minus infinite propinqua; 'Evolutio vel Longitudo lineæ Curvæ; Cen-'trum gravitatis Figuræ; Solida ejus rota-'tione genita, & eorum Superficies; sine 'ulla Radicum Extractione obtineri queant.

Ubi obiter notemus in hac Collinsii *historiola, methodum argumentandi usurpatam a D.* Newtono *in Tractatu suo* De Quadratura Curvarum, *quasi digito monstrari, dum dicit hanc Methodum exhibere Quadraturam Figurarum accuratam, si modo fieri possit, sin minus in infinitum approximan-tem,*

† In *September* 1668, Mr. *Mercator* publish'd his Logarithmotechnia, containing a Specimen of this Method, (that is, of Infinite Series) in one only Figure, to wit in the Quadrature of the Hyperbola. Not long after the Book came out, I sent one of them to Dr. *Wallis* at *Oxford*, who forthwith gave his sense of it in the Philos. Transactions: another of them I sent to Dr. *Barrow* at *Cambridge*, who forthwith sent me up some Papers of Mr. *Newton*, who is since become the Doctor's Successor in the Mathematical Lectures there. By which, and former communications made thereof from the Author to the Doctor, it appears that the said Method was invented some Years before by the said Mr. *Newton*, and generally applied: so that thereby in any Curvilinear Figure propos'd, that is determin'd by one or more common Properties, by the same Method may be obtain'd the Quadrature or Area of the said Figure, accurately when it can be done, but always infinitely near; the Evolution, or Length of the said Curve Line; the Centre of Gravity of the Figure; its round Solids made by Rotation, and their Surfaces; and all perform'd without any Extraction of Roots.

tem, atque ista omnia fieri sine ulla Extractione Radicum; Hæc enim ipsa est argumentatio in dicto Tractatu observata: & propterea hanc Methodum saltem Anno 1672 *coætaneam extitisse non est dubitandum.*

Inveni etiam in exemplari Epistolæ, a D. Collinsio *ad* D. Davidem Gregorium *prædicti* Jacobi *fratrem, datæ* 11. Aug. 1676, *præter eadem fere quæ* D. Strode *scripserat, etiam verba sequentia:*

* Supradicta Serierum Infinitarum doʻctrina, a *Newtono* biennium ante excogiʻtata fuit, quam ederetur *Mercatoris* Loʻgarithmotechnia, & generaliter omnibus ʻFiguris applicata.

Simulque affirmat se olim cum quibusdam Academicis Parisiensibus *hæc eadem scripto communicasse.*

Quapropter, cum D. Mercator *Librum suum Anno* 1668 *in lucem ediderit, sequitur eandem Doctrinam Infinitarum Serierum Figuris omnibus generaliter applicatam fuisse Anno* 1666.

Denique in Epistola, circa idem tempus ad D. Oldenburgum *scripta, asserit* Collinsius, *Jacobum Gregorium* non nisi conspecta aliqua e Seriebus

* The said Doctrine of Infinite Series was invented by Mr. *Newton*, about two Years before the Publication of Mr. *Mercator*'s Logarithmotechnia, and generally applied to all Curves.

bus *Newtoni*, quam illi impertierat, in eandem Serierum Methodum incidisse.

Ex Newtoni *autem Schedis quibusdam a me visis intellexi, quod is Quadraturam Circuli, Hyperbolæ, & aliarum quarundam Curvarum per Series Infinitas ex* Wallisii *nostri* Arithmetica Infinitorum, *per Interpolationem Serierum ejus, primo deduxit, idque Anno* 1665; *deinde Methodum excogitavit easdem Series per Divisiones & Extractiones Radicum inveniendi, quam Anno sequente generalem reddidit.*

Et cum scriptus fuerit hic Tractatus, quo tempore hæc recens inventa essent, ideo Cl. Auctor dignatus est multa in eo dilucidare, ad Re*solutionem Æquationum per Infinitas Series spectantia, quæ in aliis Libris frustra quæras.*

His subjunximus diversa Epistolarum Auctoris fragmenta, quæ ad easdem Doctrinas pertinent, quæq; olim inter Cl. Wallisii *Opera in lucem prodiere. Epistolas haud dedi integras, ut evitarem repetitionem non necessariam multarum rerum, quæ supra in Tractatu* De Analysi per Æquationes Infinitas *traditæ sunt: Quinimo Exempla quædam in iis Epistolis prætermisi ipsius Cl. Auctoris monitu, Regulas suas per se satis claras esse credentis, neque ullam dilucidationem desiderare.*

PRÆFATIO.

Inter D. Collinsii *schedas reperi autographum Epistolæ datæ* 8. Novemb. 1676, *cujus fragmentum sub finem adjeci, & dignum luce putavi; quoniam in eo memoratur solutio Problematis admodum generalis in Comparatione Curvarum usûs, quæ perficitur in Cor.* 2. *Prop.* 10. *Tract.* De Quadratura Curvarum: *Unde Lector intelligat solutionem illam Auctori jam tum innotuisse.*

Hisce Fragmentis annexus est ille ipse Tractatus De Quadratura Curvarum; *una cum altero* De Enumeratione Linearum Tertii Ordinis, *quorum uterque primum typis mandatus est Anno* 1704, *ad finem* Optices *eximiæ ejusdem Cl. Auctoris.*

Coronidis loco subjicitur Tractatulus, cui titulo est, Methodus Differentialis, *quem Cl. Auctoris permissu ex ejus autographo descripsi; Complectitur autem Doctrinam describendi Curvas ex datis Differentiis differentiarum Ordinatarum. Hæc* Methodus Differentialis *innititur Problemati ducendi Curvam Parabolici generis per data quotcunque puncta; de quo Cl. Auctor olim mentionem fecerat in Epistola sua ad D.* Oldenburgum 1676 *missa; & cujus solutionem dedit in* Lem. 5. *Lib.* 3. Princip. Philos. *non tamen prorsus eandem quoad Constructionem cum ea quam impræsentiarum tradimus.*

Hujus

PRÆFATIO.

Hujus Geometriæ Newtonianæ *non minimam esse laudem duco, quod dum per limites* Rationum Primarum *&* Ultimarum *argumentatur, æque demonstrationibus Apodicticis ac illa Veterum munitur; utpote quæ haud innititur duriusculæ illi Hypothesi quantitatum Infinite parvarum vel Indivisibilium, quarum Evanescentia obstat quominus eas tanquam quantitates speculemur. Neque parum præcellere videtur, quod tam paucis innixa Propositionibus tam late sese extendat hæc Mathesis, intra se omnia Problemata difficiliora vulgatas Methodos eludentia complectens; siquidem quicquid proponitur poterit Geometrice per alicujus Curvæ Aream effingi; vel per Methodum Universalem Extrahendi Radices ex Æquatione quavis erui; vel ad summum, ducendo Curvam per terminos quantitatum datarum solvi.*

W. Jones.

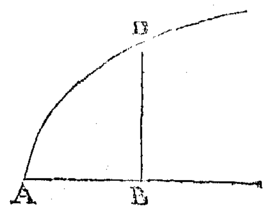
INDEX OPUSCULORUM

Quæ in hoc Libro continentur.

DE ANALYSI
Per Æquationes Numero Terminorum INFINITAS.

Methodum generalem, quam de Curvarum quantitate per Infinitam terminorum Seriem mensuranda, olim excogitaveram, in sequentibus breviter explicatam potius quam accuratè demonstratam habes.

BASI AB Curvæ alicujus AD, sit Applicata BD perpendicularis: Et vocetur $AB = x$, $BD = y$, & sint a, b, c, &c. Quantitates datæ, & m, n, Numeri Integri. Deinde,



Curvarum Simplicium Quadratura.

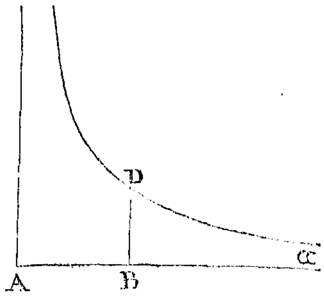
REGULA I.

Si $ax^{\frac{m}{n}} = y$; *Erit* $\frac{an}{m+n}x^{\frac{m+n}{n}}$ = *Areæ ABD.*

Res Exemplo patebit.

1. Si x^2 $(= 1x^{\frac{2}{1}}) = y$, hoc est, $a = 1 = n$, & $m = 2$; Erit $\frac{1}{3}x^3$ = ABD.

2. Si $4\sqrt{x}\ (=4x^{\frac{1}{2}})=y$; Erit $\frac{8}{3}x^{\frac{3}{2}}\ (=\frac{8}{3}\sqrt{x^3}) = ABD$.
3. Si $\sqrt[3]{x^5}\ (=x^{\frac{5}{3}})=y$; Erit $\frac{3}{8}x^{\frac{8}{3}}\ (=\frac{3}{8}\sqrt[3]{x^8}) = ABD$.
4. Si $\frac{1}{x^2}\ (=x^{-2})=y$, id est, si $a=1=n$, & $m=-2$;
Erit $(\frac{1}{-1}x^{-\frac{1}{1}}=)\ -x^{-1}\ (=\frac{-1}{x}) = \alpha BD$, infinite versus α protensæ, quam Calculus ponit negativam, propterea quod jacet ex altera parte Lineæ BD.
5. Si $\frac{1}{\sqrt{x^3}}\ (=x^{-\frac{3}{2}})=y$; Erit $(\frac{2}{-1}x^{-\frac{1}{2}}=)\ \frac{2}{-\sqrt{x}}$ $= BD\alpha$.



6. Si $\frac{1}{x}\ (=x^{-1})=y$; Erit $\frac{1}{0}x^{\frac{0}{1}}=\frac{1}{0}x^0=$ $\frac{1}{0}\times 1=\frac{1}{0}=$ Infinitæ, qualis est Area Hyperbolæ ex utraque parte Lineæ BD.

Compositarum Curvarum Quadratura ex Simplicibus.

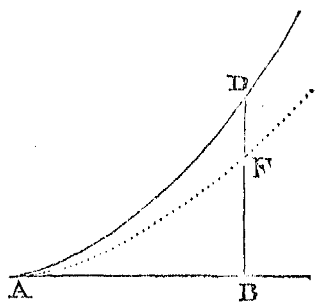
REGULA II.

Si valor ipsius y *ex pluribus istiusmodi Terminis componitur, Area etiam componetur ex Areis quæ a singulis Terminis emanant.*

Exempla Prima.

Si $x^2+x^{\frac{3}{2}}=y$; Erit $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=ABD$.

Etenim si semper sit $x^2=BF$, et $x^{\frac{3}{2}}=FD$, erit, ex præcedente Regula, $\frac{1}{3}x^3=$ superficiei AFB descriptæ per Lineam BF, et $\frac{2}{5}x^{\frac{5}{2}}=$ AFD descriptæ per DF; Quare $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=$ toti ABD.



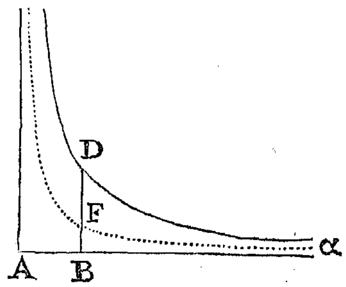
Sic si $x^2-x^{\frac{3}{2}}=y$; Erit $\frac{1}{3}x^3-\frac{2}{5}x^{\frac{5}{2}}=ABD$.

Et si $3x-2x^2+x^3-5x^4=y$; Erit $\frac{3}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4-x^5=ABD$.

Ex-

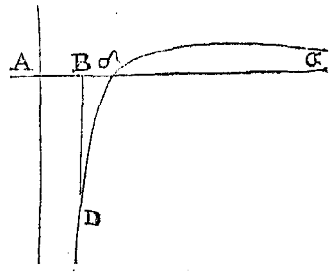
Exempla Secunda.

Si $x^{-2} + x^{-\frac{3}{2}} = y$; Erit $-x^{-1} - 2x^{-\frac{1}{2}} = \alpha BD$. Vel si $x^{-2} - x^{-\frac{3}{2}} = y$; Erit $-x^{-1} + 2x^{-\frac{1}{2}} = \alpha BD$.



Quarum signa si mutaveris habebis *Affirmativum valorem* ($x^{-1} + 2x^{-\frac{1}{2}}$ vel $x^{-1} - 2x^{-\frac{1}{2}}$) superficiei αBD, modo tota cadat supra basim $AB\alpha$.

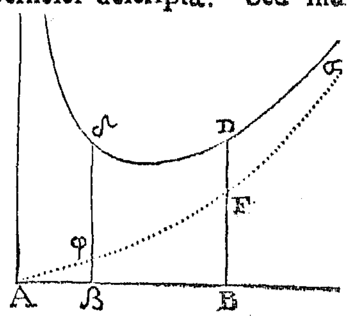
Sin aliqua pars cadat infra (quod fit cum Curva decussat suam Basin inter B et α, ut hic vides in δ,) ista parte a parte superiori subducta, habebis valorem Differentiæ : Earum vero Summam si cupis, quære utramque Superficiem seorsim, et adde. Quod idem in reliquis hujus Regulæ exemplis notandum volo.



Exempla Tertia.

Si $x^2 + x^{-2} = y$; Erit $\frac{1}{3}x^3 - x^{-1} =$ Superficiei descriptæ. Sed hic notandum est, quod dictæ Superficei partes sic inventæ jacent ex diverso latere Lineæ BD.

Nempe, posito $x^2 = BF$, & $x^{-2} = FD$; Erit $\frac{1}{3}x^3 = ABF$ Superficiei per BF descriptæ, & $-x^{-1} = DF\alpha$ Superficiei descriptæ per DF.



Et

Et hoc semper accidit cum Indices ($\frac{m+n}{n}$) rationum Basis x in valore Superficiei quæsitæ, sint variis Signis affecti. In hujusmodi Casibus, pars aliqua $BD\delta\beta$ Superficiei media (quæ sola dari poterit, cum Superficies sit utrinque infinita) sic invenitur.

Subtrahe Superficiem ad minorem Basin $A\beta$ pertinentem, a Superficie ad majorem Basin AB pertinente, & habebis $\beta BD\delta$ Superficiem differentiæ Basium insistentem. Sic in hoc Exemplo. (*Vide Fig. Præcedentem.*)

Si $AB = 2$, & $A\beta = 1$; Erit $\beta BD\delta = \frac{17}{6}$:

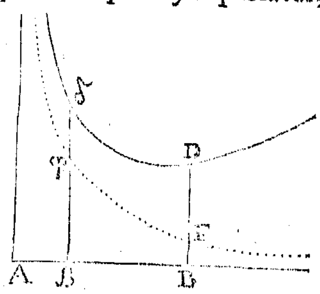
Etenim Superficies ad AB pertinens (viz. $ABF - DF\alpha$) erit $\frac{8}{3} - \frac{1}{2}$ sive $\frac{13}{6}$; et superficies ad $A\beta$ pertinens (viz. $A\varphi\beta - \delta\varphi\alpha$) erit $\frac{1}{3} - 1$, sive $-\frac{2}{3}$: et earum differentia (viz. $ABF - DF\alpha - A\varphi\beta + \delta\varphi\alpha = \beta BD\delta$) erit $\frac{13}{6} + \frac{2}{3}$ sive $\frac{17}{6}$.

Eodem modo, si $A\beta = 1$, $AB = x$; Erit $\beta BD\delta = \frac{2}{3} + \frac{1}{3}x^3 - x^{-1}$.

Sic si $2x^3 - 3x^5 - \frac{2}{3}x^{-4} + x^{-\frac{3}{5}} = y$, & $A\beta = 1$;

Erit $\beta BD\delta = \frac{1}{2}x^4 - \frac{1}{2}x^6 + \frac{2}{9}x^{-3} + \frac{5}{2}x^{\frac{2}{5}} - \frac{49}{18}$.

Denique notari poterit quod si quantitas x^{-1} in valore ipsius y reperiatur, iste Terminus (cum Hyperbolicam superficiem generat) seorsim a reliqius considerandus est.



Ut si $x^2 + x^{-3} + x^{-1} = y$: Sit $x^{-1} = BF$, & $x^2 + x^{-3} = FD$, ac $A\beta = 1$; Et erit $\delta\varphi FD = \frac{1}{6} + \frac{1}{3}x^3 - \frac{1}{2}x^{-2}$, utpote quæ ex Terminis $x^2 + x^{-3}$ generatur.

Quare, si reliqua Superficies $\beta\varphi FB$, quæ Hyperbolica est, ex Calculo aliquo sit data, dabitur tota $\beta BD\delta$.

Aliarum

Aliarum Omnium Quadratura.

REGULA III.

Sin valor ipsius y, *vel aliquis ejus Terminus sit præcedentibus magis compositus, in Terminos Simpliciores reducendus est; operando in Literis ad eundem Modum quo Arithmetici in Numeris Decimalibus dividunt, Radices extrahunt, vel affectas Æquationes solvunt; & ex istis Terminis quæsitam Curvæ Superficiem, per præcedentes Regulas deinceps elicies.*

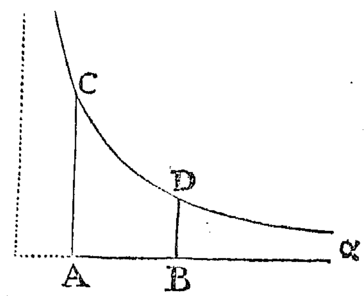
Exempla Dividendo.

Sit $\frac{aa}{b+x} = y$; Curva nempe existente Hyperbola.

Jam ut Æquatio ista a Denominatore suo liberetur, Divisionem sic instituo.

$$b+x)\ aa + 0\ \left(\frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4}\ \&c.\right.$$

$$\begin{array}{l} aa + \frac{aax}{b} \\ \hline 0 - \frac{aax}{b} + 0 \\ \quad - \frac{aax}{b} - \frac{aax^2}{b^2} \\ \hline \qquad 0 + \frac{aax^2}{b^2} + 0 \\ \qquad + \frac{aax^2}{b^2} + \frac{aax^3}{b^3} \\ \hline \qquad\quad 0 - \frac{aax^3}{b^3} + 0 \\ \qquad\quad - \frac{aax^3}{b^3} - \frac{aax^4}{b^4} \\ \hline \qquad\qquad 0 + \frac{aax^4}{b^4} \\ \qquad\qquad \&c. \end{array}$$



Et sic vice hujus $y = \frac{aa}{b+x}$, nova prodit $y = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4}$, &c. serie istac infinite continuata; Adeoque (per Regulam Secundam) Area quæsita ABDC æqualis erit ipsi $\frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}$ &c. infinitæ etiam seriei, cujus tamen Termini pauci initiales sunt in usum quemvis satis exacti, si modo x sit aliquoties minor quam b.

Eodem modo, si sit $\frac{1}{1+xx} = y$, Dividendo prodibit $y = 1 - x^2 + x^4 - x^6 + x^8$ &c. Unde (per Regulam Secundam) erit ABDC $= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$ &c.

Vel si Terminus xx ponatur in divisore primus, hoc modo $xx + 1$), prodibit $x^{-2} - x^{-4} + x^{-6} - x^{-8}$ &c. pro valore ipsius y; Unde (per Regulam Secundam) erit $BD\alpha = -x^{-1} + \frac{1}{3}x^{-3} - \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7}$ &c. Priori modo procede cum x est satis parva, posteriori cum satis magna supponitur.

Denique si $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = y$; Dividendo prodit

$2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$ &c. unde erit

ABDC $= \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{3}x^3$ &c.

Exempla Radicem Extrahendo.

Si sit $\sqrt{aa+xx} = y$, Radicem sic extraho,

$$aa + xx \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \text{ \&c.} \right.$$

$$\underline{aa}$$

$$0 + xx$$

$$\underline{xx + \frac{x^4}{4a^2}}$$

$$0 - \frac{x^4}{4a^2}$$

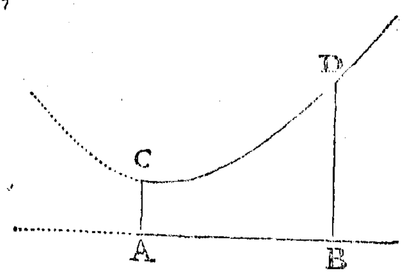
$$\underline{-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}}$$

$$0 + \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

$$\underline{+\frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}}}$$

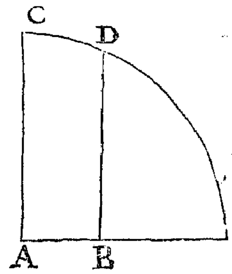
$$0 - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}$$

&c.



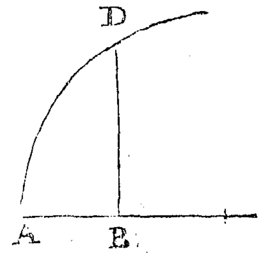
Unde,

Unde, pro Æquatione $\sqrt{aa+xx}=y$, nova producitur, viz. $y = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. Et (per Reg. 2.) Area quæsita ABDC erit $= ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. Et hæc est Quadratura Hyperbolæ.



Eodem modo, si sit $\sqrt{aa-xx}=y$, ejus Radix erit $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. Adeoque Area quæsita ABDC erit æqualis $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. Et hæc est Quadratura Circuli.

Vel si ponas $\sqrt{x-xx}=y$; erit Radix æqualis infinitæ seriei $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}}$ &c. Et Area quæsita ABD æqualis erit $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}}$ &c. sive $x^{\frac{1}{2}}$ in $\frac{2}{3}x - \frac{1}{5}x^2 - \frac{1}{28}x^3 - \frac{1}{72}x^4 - \frac{5}{704}x^5$ &c. Et hæc est Areæ Circuli Quadratura.



Si $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}} = y$, (Cujus Quadratura dat Longitudinem curvæ Ellipticæ;) Extrahendo radicem utramq; prodit

$$\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6 - \frac{5}{128}a^4x^8}{1 - \frac{1}{2}bx^2 - \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6 - \frac{5}{128}b^4x^8} \text{ \&c.}$$

Et Dividendo, sicut fit in Fractionibus Decimalibus, habes

$$\begin{array}{lllll}
1 + \frac{1}{2}bx^2 & + \frac{3}{8}b^2x^4 & + \frac{5}{16}b^3x^6 & + \frac{35}{128}b^4x^8 & \text{\&c.} \\
\quad + \frac{1}{2}a & + \frac{1}{4}ab & + \frac{3}{16}ab^2 & + \frac{5}{32}ab^3 & \\
 & - \frac{1}{8}a^2 & - \frac{1}{16}a^2b & - \frac{3}{64}a^2b^2 & \\
 & & + \frac{1}{16}a^3 & + \frac{1}{32}a^3b & \\
 & & & - \frac{5}{128}a^4 &
\end{array}$$

Adeoque Aream quæsitam

$$\begin{array}{lll}
x + \frac{1}{6}bx^3 & + \frac{3}{40}b^2x^5 & \text{\&c.} \\
\quad + \frac{1}{6}a & + \frac{1}{20}ab & \\
 & - \frac{1}{40}a^2 &
\end{array}$$

Sed

Sed obſervandum eſt, quod Operatio non raro abbreviatur per debitam Æquationis præparationem, ut in allato Exemplo $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}} = y$. Si utramque partem fractionis per $\sqrt{1-bxx}$ multiplices prodibit

$\frac{\sqrt{1 \begin{smallmatrix}+a\\-b\end{smallmatrix} x^2 - abx^4}}{1+bx^2} = y$, & reliquum opus perficitur extrahendo Radicem Numeratoris tantum, & dividendo per Denominatorem.

Ex hiſce, credo, ſatis patebit modus reducendi quemlibet valorem ipſius y (quibuſcunque Radicibus vel Denominatoribus ſit perplexus, ut hic videre eſt;

$$x^3 + \frac{\sqrt{x - \sqrt{1-xx}}}{\sqrt[3]{ax^2+x^3}} - \frac{\sqrt[5]{x^3+2x^5-x^{\frac{3}{2}}}}{\sqrt[3]{x+x^2} - \sqrt[2]{2x-x^{\frac{2}{3}}}} = y) \text{ in ſeries Infinitas}$$

ſimplicium Terminorum, ex quibus, per Regulam Secundam, quæſita Superficies cognoſcetur.

Exempla per Reſolutionem Æquationum.

NUMERALIS ÆQUATIONUM AFFECTARUM RESOLUTIO.

Quia tota difficultas in Reſolutione latet, modum quo ego utor in Æquatione Numerali primum illuſtrabo.

Sit $y^3 - 2y - 5 = 0$, reſolvenda : Et ſit 2, numerus qui minus quam decima ſui parte differt a Radice quæſita. Tum pono $2 + p = y$, & ſubſtituo hunc ipſi valorem in Æquationem, & inde nova prodit $p^3 + 6p^2 + 10p - 1 = 0$, cujus Radix p exquirenda eſt, ut quotienti addatur : Nempe (neglectis $p^3 + 6p^2$ ob parvitatem) $10p - 1 = 0$, ſive $p = 0{,}1$ prope veritatem eſt; itaque ſcribo 0,1 in quotiente, & ſuppono $0{,}1 + q = p$, & hunc ejus valorem, ut prius ſubſtituo, unde prodit $q^3 + 6{,}3q^2 + 11{,}23q + 0{,}061 = 0$.

Et cum $11{,}23q + 0{,}061$ veritati prope accedit, ſive fere ſit q æqualis $-0{,}0054$ (dividendo nempe donec tot eliciantur Figuræ, quot locis primæ Figuræ hujus & principalis quotientis excluſive diſtant) ſcribo $-0{,}0054$ in inferiori parte quotientis, cum negativa ſit.

Et

Et ſupponens $-0,0054+r=q$, hunc ut prius ſubſtituo, & operationem ſic produco quo uſq; placuerit. Verum ſi ad bis tot figuras tantum quot in Quotiente jam reperiuntur una dempta, operam continuare cupiam, pro q ſubſtituo $-0,0054+r$ in hanc $6,3q^2+11,23q+0,061$, ſcilicet primo ejus termino (q^3) propter exilitatem ſuam

$y^3-2y-5=0$		$+2,10000000$ $-0,00544853$ $+2,09455147=y$
$2+p=y$	$+y^3$	$+8+12p+6p^2+p^3$
	$+2y$	$-4-2p$
	-5	-5
	Summa	$-1+10p+6p^2+p^3$
$0,1+q=p$	$+p^3$	$+0,001+0,03q+0,3q^2+q^3$
	$+6p^2$	$+0,06 \quad +1,2 \quad +6,0$
	$+10p$	$+1, \quad +10,$
	-1	$-1,$
	Summa	$+0,061+11,23q+6,3q^2+q^3$
$-0,0054+r=q$	$+6,3q^2$	$+0,000183708-0,06804r+6,3r^2$
	$+11,23q$	$-0,060642 \quad +11,23$
	$+0,061$	$+0,061$
	Summa	$+0,000541708+11,16196r+6,3r^2$
$-0,00004854+s=r$		

neglecto, & prodit $6,3r^2+11,16196r+0,000541708=0$ fere, ſive (rejecto $6,3r^2$) $r=\frac{-0,000541708}{11,16196}=-0,00004853$ fere, quam ſcribo in negativa parte Quotientis. Denique negativam partem Quotientis ab Affirmativa ſubducens habeo 2,09455147 Quotientem quæſitam.

Æquationes plurium dimenſionum nihilo ſecius reſolvuntur, & operam ſub fine, ut hic factum fuit levabis, ſi primos ejus terminos gradatim omiſeris.

Præterea notandum eſt quod in hoc exemplo, ſi dubitarem an $0,1=p$ veritati ſatis accederet, pro $10p-1=0$, finxiſſem $6p^2+10p-1=0$, & ejus radicis primam figuram in Quotiente ſcripſiſſem; & ſecundam vel tertiam Quotientis figuram ſic explorare convenit, ubi in Æquatione iſta ultimo reſultante quadratum coefficientis penultimi termini, non ſit decies majus quam factus ex ultimo termino ducto in coefficientem termini antepenultimi.

Imo laborem plerumque minues præsertim in Æquationibus plurimarum dimensionum, si figuras omnes Quotienti addendas dicto modo (hoc est extrahendo minorem radicum, ex tribus ultimis terminis Æquationis novissime resultantis) exquiras: Isto enim modo figuras duplo plures qualibet vice Quotienti lucraberis.

Hæc Methodus resolvendi Æquationes pervulgata an sit nescio, certe mihi videtur præ reliquis simplex, & usui accommodata. Demonstratio ejus ex ipso modo operandi patet, unde cum opus sit, in memoriam facile revocatur.

Æquationes in quibus vel aliqui vel nulli Termini desint, eadem fere facilitate tractantur; & Æquatio semper relinquitur, cujus Radix una cum acquisita Quotiente adæquat Radicem Æquationis primo propositæ. Unde Examinatio Operis hic æque poterit institui ac in reliqua Arithmetica, auferendo nempe Quotientem a Radice primæ Æquationis (sicut Analystis notum est) ut Æquatio ultima vel Termini ejus duo tresve ultimi producantur inde. Quicquid laboris hic est, istud in Operatione substituendi quantitates unas pro aliis reperietur: Id quod varie perficias, at sequentem modum maxime expeditum puto, præsertim ubi Numeri Coefficientes constant ex pluribus Figuris.

Sit $p+3$ substituenda pro y in hanc $y^4-4y^3+5y^2-12y+17=0$. Et cum ista possit resolvi in hanc formam

$\overline{\overline{\overline{y-4}\times y+5}\times y-12}\times y+17=0$. Æquatio nova sic generabitur $\overline{p-1}\times\overline{p+3}=p^2+2p-3$. et p^2+2p+2 in $p+3=p^3+5p^2+8p+6$. et p^3+5p^2+8p-6 in $p+3=p^4+8p^3+23p^2+18p-18$. et $p^4+8p^3+23p^2+18p-1=0$, quæ quærebatur.

LITERALIS ÆQUATIONUM AFFECTARUM RESOLUTIO.

His in numeris sic ostensis: Sit Æquatio literalis $y^3+a^2y-2a^3+axy-x^3=0$, resolvenda.

Primum inquiro valorem ipsius y cum x sit nulla, hoc est, elicio Radicem hujus Æquationis $y^3+a^2y-2a^3=0$, & invenio esse $+a$. Itaque scribo $+a$ in Quotiente, & supponens $+a+p=y$, substituo pro y valorem ejus, & Terminos inde resultantes ($p^3+3ap^2+4a^2p$, &c.) margini appono; Ex quibus assumo $+4a^2p+a^2x$ terminos utique ubi p & x seorsim sunt minimarum dimensionum, & eos nihilo fere æquales esse suppono, sive $p=-\frac{1}{4}x$ fere, vel $p=-\frac{1}{4}x+q$. Et scribens

$-\frac{1}{4}x$

$-\frac{1}{4}x$ in Quotiente, ſubſtituo $-\frac{1}{4}x+q$ pro p; Et terminos inde reſultantes iterum in margine ſcribo, ut vides in annexo ſchemate, & inde aſſumo Quantitates $+4a^2q-\frac{1}{16}ax^2$, in quibus utiq; q & x ſeorſim ſunt minimarum dimenſionum, & fingo $q=\frac{xx}{64a}$ fere, ſive $q=+\frac{xx}{64a}+r$; & adnectens $+\frac{xx}{64a}$ Quotienti, ſubſtituo $\frac{xx}{64a}+r$ pro q; & ſic procedo quo uſque placuerit.

$$y^3+a^2y-2a^3+axy-x^3=0$$

$$y=a-\frac{x}{4}+\frac{x^2}{64a}-\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}\ \&c.$$

$+a+p=y$	$+y^3$	$+a^3+3a^2p+3ap^2+p^3$
	$+a^2y$	$+a^3+a^2p$
	$+axy$	$+a^2x+axp$
	$-2a^3$	$-2a^3$
	$-x^3$	$-x^3$
$-\frac{1}{4}x+q=p$	$+p^3$	$-\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$
	$+3ap^2$	$+\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2$
	$+4a^2p$	$-a^2x+4a^2q$
	$+axp$	$-\frac{1}{4}ax^2+axq$
	$+a^2x$	$+a^2x$
	$-x^3$	$-x^3$
$+\frac{x^2}{64a}+r=q$	$+3aq^2$	$+\frac{3x^4}{4096a}+\frac{3}{32}x^2r+3ar^2$
	$+4a^2q$	$+\frac{1}{16}ax^2+4a^2r$
	$-\frac{1}{2}axq$	$-\frac{1}{128}x^3-\frac{1}{2}axr$
	$+\frac{3}{16}x^2q$	$-\frac{3x^4}{1024a}+\frac{3}{16}x^2r$
	$-\frac{1}{16}ax^2$	$-\frac{1}{16}ax^2$
	$-\frac{65}{64}x^3$	$-\frac{65}{64}x^3$

$$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}\left(+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}\right.$$

Sin duplo tantum plures Quotienti terminos, uno dempto, jungendos adhuc vellem: Primo termino (q^3) Æquationis noviſſime reſultantis miſſo, & iſta etiam parte ($-\frac{3}{4}xq^2$) ſecundi, ubi x eſt tot dimenſionum quot in penultimo termino Quotientis; In reliquos terminos ($3aq^2+4a^2q$, &c.)

margini

margini adſcriptos ut vides, ſubſtituo $\frac{x^2}{64a} + r$ pro q; & ex ultimis duobus terminis ($\frac{15x^4}{4096a} - \frac{131}{128}x^3 + \frac{9}{32}x^2r - \frac{1}{2}axr + 4a^2r$) Æquationis inde reſultantis, facta diviſione $4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2$) $+ \frac{131}{128}x^3 - \frac{15x^4}{4096a}$ (elicio $+ \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ Quotienti adnectendos.

Denique Quotiens iſta ($a - \frac{x}{4} + \frac{xx}{64a}$, &c.) per Regulam ſecundam, dabit $ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \frac{509x^5}{81920a^3}$, &c. pro Area quæſita, quæ ad veritatem tanto magis accedit, quanto x ſit minor.

Alius modus eaſdem Reſolvendi.

Sin valor Areæ tanto magis ad veritatem accedere debet quanto x ſit major; Exemplum eſto $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. Itaque hanc reſoluturus excerpo terminos $y^3 + x^2y - 2x^3$ in quibus x & y vel ſeorſim, vel ſimul multiplicatæ, ſunt & plurimarum, & æqualium ubique dimenſionum; & ex iis quaſi nihilo æqualibus Radicem elicio. Hanc invenio eſſe x, & in Quotiente ſcribo. Vel quod eodem recidit, ex $y^3 + y - 2$ (unitate pro x ſubſtituta) Radicem extraho quæ hic prodit 1, & eam per x multiplico, & factum (x) in Quotiente ſcribo. Denique pono $x + p = y$, & ſic procedo ut in priori Exemplo, donec habeam Quotientem $x - \frac{a}{4} + \frac{aa}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$, &c. adeoque Aream $\frac{x^2}{2} - \frac{ax}{4} + \boxed{\frac{aa}{64x}} - \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$, de qua vide exempla tertia Regulæ ſecundæ. Lucis gratia dedi hoc exemplum in omnibus idem cum priori, modo x & a ſibi invicem ibi ſubſtituantur, ut non opus eſſet aliud Reſolutionis exemplum hic adjungere.

Area autem ($\frac{xx}{2} - \frac{ax}{4} + \boxed{\frac{aa}{64x}}$ &c. terminatur ad Curvam quæ juxta Aſymptoton aliquam in infinitum ſerpit; & Termini initiales ($x - \frac{1}{4}a$) valoris extracti de y, in Aſymptoton iſtam ſemper terminantur; unde portionem Aſymptoti facile invenies. Idem ſemper notandum eſt cum Area deſignatur terminis plus pluſque diviſis per x continue, præterquam quod vice Aſymptoti rectæ quandoque habeatur Parabola Conica, vel alia magis compoſita.

Sed

Sed hunc modum miſſum faciens, utpote particularem, quia non applicabilem Curvis in orbem ad inſtar Ellipſium flexis; de altero modo per exemplum $y^3+a^2y+axy-2a^3-x^3=0$, ſupra oſtenſo (ſcilicet quo demenſiones ipſius x in numeratoribus quotientis perpetuo augeantur) annotabo ſequentia.

1. Si quando accidit quod valor ipſius y, cum x nullum eſſe fingitur, ſit quantitas ſurda vel penitus ignota, licebit illam litera aliqua deſignare. Ut in exemplo, $y^3+a^2y+axy-2a^3-x^3=0$, ſi radix hujus $y^3+a^2y-2a^3$ fuiſſet ſurda vel ignota, finxiſſem quamlibet (b) pro ea ponendam; et reſolutionem ut ſequitur perfeciſſem. Scribens b in Quotiente, ſuppono $b+p=y$, & iſtum pro y ſubſtituo, ut vides; unde nova p^3+3bp^2, &c. reſultat, rejectis terminis $b^3+a^2b-2a^3$, qui nihilo ſunt æquales, propterea quod b ſupponitur Radix hujus $y^3+a^2y-2a^3=0$. Deinde termini $3b^2p+a^2p+abx$ dant $-\frac{abx}{3b^2+a^2}$ quotienti apponendum, et $-\frac{abx}{3b^2+a^2}+q$ ſubſtituendum pro p, &c.

$$y^3+aay+axy-2a^3-x^3=0. \quad \text{Sit } cc=3b^2+a^2.$$

$$y=b-\frac{abx}{c^2}+\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}-\frac{a^5bx^3}{c^8}+\frac{a^5b^3x^3}{c^{10}}\ \&c.$$

$b+p=y$	y^3	$+b^3+3b^2p+3bp^2+p^3$
	$+axy$	$+abx+axp$
	$+aay$	$+aab+aap$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$\frac{-abx}{cc}+q=p$	p^3	$-\frac{a^3b^3x^3}{c^6}$ &c.
	$+3bp^2$	$+\frac{3a^2b^3x^2}{c^4}-\frac{6ab^2x}{c^2}q$ &c.
	$+axp$	$-\frac{a^2bx^2}{c^2}+axq$
	$+ccp$	$-abx+ccq$
	$-x^3$	$-x^3$
	$+abx$	$+abx$

$$c^2+ax-\frac{6ab^2x}{cc}\Big)\ \frac{a^4bx^2}{c^4}+x^3+\frac{a^3b^3x^3}{c^6}\ \Big(\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}\ \&c.$$

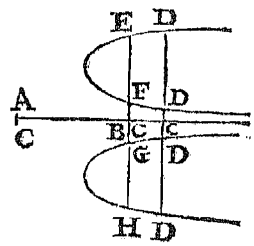
Completo opere, ſumo numerum aliquem pro a, & hanc $y^3+a^2y-2a^3=0$, ſicut de numerali æquatione oſtenſum ſupra reſolvo; & radicem ejus pro b ſubſtituo.

2. Si dictus valor ſit nihil, hoc eſt ſi in æquatione reſolvenda nullus ſit terminus niſi qui per x vel y ſit multiplicatus, ut in hac $y^3-axy+x^3=0$; tum terminos $(-axy+x^3)$ ſeligo in quibus x ſeorſim & y etiam ſeorſim ſi fieri poteſt, alias per x multiplicata, ſit minimarum dimenſionum. Et

 illi

illi dant $+\frac{xx}{a}$ pro primo termino quotientis, & $\frac{xx}{a}+p$ pro y ſubſtituendum. In hac $y^3-a^2y+axy-x^3=0$, licebit primum terminum quotientis vel ex $-a^2y-x^3$, vel ex y^3-a^2y elicere.

3. Si valor iſte ſit imaginarius, ut in hac $y^4+y^2-2y+6-x^2y^2-2x+x^2+x^4=0$, augeo vel imminuo quantitatem x donec dictus valor evadat realis.



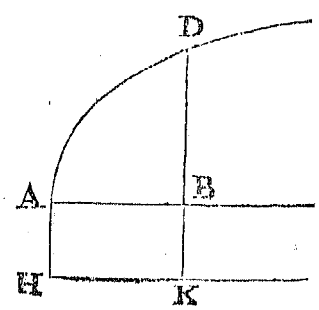
Sic in annexo ſchemate, cum AC (x) nulla eſt, tum CD (y) eſt imaginaria.

Sin minuatur AC per datam AB, ut BC fiat x; tum poſito quod BC (x) ſit nulla, CD (y) erit valore quadruplici (CE, CF, CG, vel CH) realis; quarum radicum (CE, CF, CG, vel CH) quælibet poteſt eſſe primus terminus quotientis, prout ſuperficies BEDC, BFDC, BGDC, vel BHDC deſideratur. In aliis etiam caſibus, ſi quando hæſitas, te hoc modo extricabis.

Deniq; ſi index poteſtatis ipſius x vel y ſit fractio, reduco ipſum ad integrum: ut in hoc exemplo $y^3-xy^{\frac{1}{2}}+x^{\frac{4}{3}}=0$. Poſito $y^{\frac{1}{2}}=v$, & $x^{\frac{1}{3}}=z$, reſultabit $v^6-z^3v+z^4=0$, cujus radix eſt $v=z+z^3$, &c. ſive (reſtituendo valores) $y^{\frac{1}{2}}=x^{\frac{1}{3}}+x$, &c. & quadrando $y=x^{\frac{2}{3}}+2x^{\frac{4}{3}}$, &c.

Et hæc de areis curvarum inveſtigandis dicta ſufficiant. Imo cum Problemata omnia de curvarum Longitudine, de quantitate & ſuperficie ſolidorum, deque Centro Gravitatis, poſſunt eo tandem reduci ut quæratur quantitas Superficiei planæ linea curva terminatæ, non opus eſt quicquam de iis adjungere. In iſtis autem quo ego operor modo dicam breviſſime.

Applicatio prædictorum ad reliqua iſtiuſmodi Problemata.

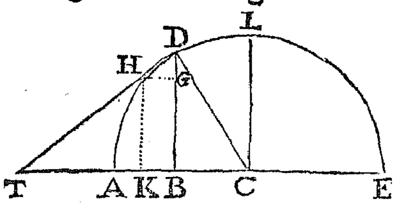


Sit ABD curva quævis, & AHKB rectangulum cujus latus AH vel BK eſt unitas. Et cogita rectam DBK uniformiter ab AH motam, areas ABD & AK deſcribere; & quod BK (1) ſit momentum quo AK (x) & BD (y) momentum quo ABD gradatim augetur; & quod ex momento BD perpetim dato, poſſis, per prædictas regulas, aream ABD ipſo deſcriptam inveſtigare, ſive cum AK (x) momento 1 deſcripta conferre.

Jam

Jam qua ratione Superficies ABD ex momento suo perpetim dato, per præcedentes regulas elicitur, eadem quælibet alia quantitas ex momento suo sic dato elicietur. Exemplo res fiet clarior.

Longitudines Curvarum invenire.



Sit ADLE circulus cujus arcus AD longitudo est indaganda. Ducto tangente DHT, & completo indefinite parvo rectangulo HGBK, & posito $AE = 1 = 2AC$. Erit ut BK sive GH, momentum Basis AB (x), ad HD momentum Arcus AD :: BT : DT :: BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) :: 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH). Adeoque $\frac{1}{2\sqrt{x-xx}}$ sive $\frac{\sqrt{x-xx}}{2x-2xx}$ est momentum Arcus AD.

Quod reductum fit $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}}$ &c.
Quare, per regulam secundam, longitudo Arcus AD est
$x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}}$ &c.
sive $x^{\frac{1}{2}}$ in $1 + \frac{1}{6}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 + \frac{63}{2816}x^5$, &c.

Non secus ponendo CB esse x, & radium CA esse 1, invenies Arcum LD esse $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

Sed notandum est quod unitas ista quæ pro momento ponitur est Superficies cum de Solidis, & linea cum de superficiebus, & punctum cum de lineis (ut in hoc exemplo) agitur.

Nec vereor loqui de unitate in punctis, sive lineis infinite parvis, siquidem proportiones ibi jam contemplantur Geometræ, dum utuntur methodis Indivisibilium.

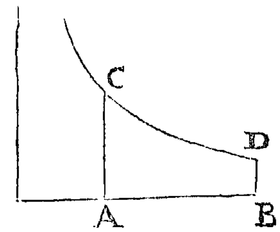
Ex his fiat conjectura de superficiebus & quantitatibus solidorum, ac de Centris Gravitatum.

Invenire prædictorum conversum.

Verum si e contra ex area vel longitudine &c. Curvæ alicujus data longitudo Basis AB desideratur, ex æquationibus per præcedentes regulas inventis extrahatur radix de x.

Inventio Basis ex Area data.

Ut si ex area ABDC Hyperbolæ ($\frac{1}{1+x} = y$) data, cupiam basim AB investigare, area ista z nominata, extraho radicem hujus z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglectis illis terminis in quibus x est plurium dimensionum quam z in quotiente desideratur.



Ut si vellem quod z ad quinque tantum dimensiones in quotiente ascendat, negligo omnes $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. et radicem hujus tantum $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$ extraho.

		$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.
$z + p = x$	$+\frac{1}{5}x^5$	$+\frac{1}{5}z^5$, &c.
	$-\frac{1}{4}x^4$	$-\frac{1}{4}z^4 - z^3p$, &c.
	$+\frac{1}{3}x^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$, &c.
	$-\frac{1}{2}x^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$.
	$+x$	$+z+p$
	$-z$	$-z$
$\frac{1}{2}z^2 + q = p$	$+zp^2$	$+\frac{1}{4}z^5$, &c.
	$-\frac{1}{2}p^2$	$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$, &c.
	$-z^3p$	$-\frac{1}{2}z^5$, &c.
	$+z^2p$	$+\frac{1}{2}z^4 + z^2q$
	$-zp$	$-\frac{1}{2}z^3 - zq$
	$+p$	$+\frac{1}{2}z^2 + q$
	$+\frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$-\frac{1}{4}z^4$	$-\frac{1}{4}z^4$
	$+\frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$-\frac{1}{2}z^2$	$-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2)$ $\frac{1}{6}z^3 - \frac{1}{8}z^4 - \frac{1}{20}z^5$ $(\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

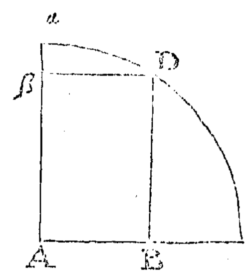
Analysin ut vides exhibui propter adnotanda duo sequentia.

1. Quod inter substituendum, istos terminos semper omitto quos nulli deinceps usui fore prævideam. Cujus rei regula esto, quod post primum terminum ex qualibet quantitate sibi collaterali resultantem non addo plures terminos dextrorsum quam istius primi termini index dimensionis ab

ab indice dimenſionis maximæ unitatibus diſtat. Ut in hoc exemplo, ubi maxima dimenſio eſt 5, omiſi omnes terminos poſt z^5, poſt z^4 poſui unicum, & duos tantum poſt z^3. Cum radix extrahenda (x) ſit parium ubique, vel imparium dimenſionum, hæc eſto regula; Quod poſt primum terminum ex qualibet quantitate ſibi collaterali reſultantem non addo plures terminos dextrorſum, quam iſtius primi termini index dimenſionis ab indice dimenſionis maximæ binis unitatibus diſtat; vel ternis unitatibus, ſi indices dimenſionum ipſius x unitatibus ubique ternis a ſe invicem diſtant, & ſic de reliquis.

2. Cum video p, q, vel r, &c. in æquatione noviſſime reſultante eſſe unius tantum dimenſionis, ejus valorem, hoc eſt, reliquos terminos quotienti addendos, per diviſionem quæro. Ut hic vides factum.

Inventio Baſis ex data Longitudine Curvæ.



Si ex dato arcu αD Sinus AB deſideratur; æquationis $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. ſupra inventæ, (poſito nempe AB $= x$, αD $= z$, & A$\alpha = 1$,) radix extracta erit $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.

Et præterea ſi Coſinum Aβ ex iſto arcu dato cupis, fac Aβ $(= \sqrt{1-xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

De Serie progreſſionum continuanda.

Hic obiter notetur, quod 5 vel 6 terminis iſtarum radicum cognitis, eas plerumque ex analogia obſervata poteris ad arbitrium producere.

Sic hanc $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. produces dividendo ultimum terminum per hos ordine numeros 2, 3, 4, 5, 6, 7, &c.

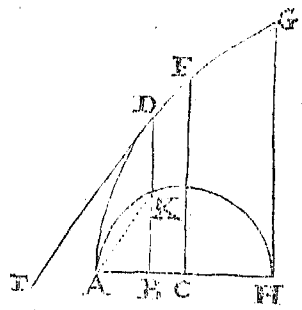
Et hanc $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, &c. per hos 2×3, 4×5, 6×7, 8×9, 10×11, &c.

Et hanc $x = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6$, &c. per hos 1×2, 3×4, 5×6, 7×8, 9×10, &c.

Et hanc $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. multiplicando per hos $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, &c. Et ſic in reliquis.

Applicatio prædictorum ad Curvas Mechanicas.

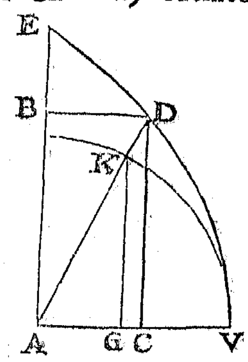
Et hæc de curvis Geometricis dicta sufficiant. Quinetiam curva etiamsi Mechanica sit, methodum tamen nostram nequaquam respuit.



Exemplo sit Trochoides, ADFG, cujus vertex A, & axis AH, & AKH rota qua describitur. Et quæratur Superficies ABD. Jam posito AB $= x$, BD $= y$, ut supra, & AH $= 1$; primo quæro Longitudinem ipsius BD. Nempe ex natura Trochoidis est KD = arcui AK. Quare tota BD = BK + arc. AK. Sed est BK $(= \sqrt{x - xx}) = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c. & (ex prædictis) arcus AK $= x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}}$, &c. Ergo tota BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}}$, &c. Et (per Reg. 2.) area ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}}$, &c.

Vel brevius sic: Cum recta AK tangenti TD parallela sit, erit AB ad BK sicut momentum lineæ AB ad momentum lineæ BD, hoc est $x : \sqrt{x - xx} :: 1 : \frac{1}{x}\sqrt{x - xx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}}$, &c. Quare (per Reg. 2.) BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} - \frac{5}{576}x^{\frac{9}{2}}$, &c. Et superficies ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} - \frac{5}{3168}x^{\frac{11}{2}}$, &c.

Non dissimili modo (posito C centro circuli, & CB $= x$) obtinebis aream CBDF, &c.



Sit area ABDV Quadratricis VDE (cujus vertex est V, & A centrum circuli interioris VK cui aptatur) invenienda. Ducta qualibet AKD, demitto perpendiculares DB, DC, KG. Eritque KG : AG :: AB (x) : BD (y), sive $\frac{x \times AG}{KG} = y$. Verum ex natura Quadratricis est BA (= DC) = arcui VK, sive VK $= x$. Quare posito AV $= 1$, erit GK $= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, &c. ex supra ostensis, & GA $= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$, &c.

Adeoque

Adeoque $y\ (= \frac{x \times AG}{KG}) = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6}$ &c. ſive, diviſione facta, $y = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6$, &c. et (per Reg. 2.) area $AVDB = x - \frac{1}{9}x^3 - \frac{1}{225}x^5 - \frac{2}{6615}x^7$, &c.

Sic longitudo Quadratricis VD, licet calculo difficiliori, determinabilis eſt.

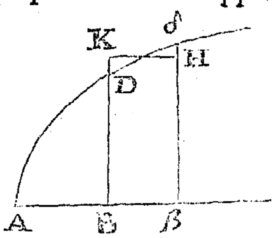
Nec quicquam hujuſmodi ſcio ad quod hæc methodus idque variis modis, ſeſe non extendit. Imo tangentes ad curvas Mechanicas (ſi quando id non alias fiat) hujus ope ducantur. Et quicquid vulgaris Analyſis per æquationes ex finito terminorum numero conſtantes (quando id ſit poſſibile) perficit, hæc per æquationes infinitas ſemper perficiat: Ut nil dubitaverim nomen Analyſis etiam huic tribuere. Ratiocinia nempe in hac non minus certa ſunt quam in illa, nec æquationes minus exactæ; licet omnes earum terminos, nos homines & rationis finitæ nec deſignare neque ita concipere poſſimus, ut quantitates inde deſideratas exacte cognoſcamus: Sicut radices ſurdæ finitarum æquationum nec numeris nec quavis arte Analytica ita poſſunt exhiberi ut alicujus quantitas a reliquis diſtincta exacte cognoſcatur.

Denique ad Analyticam merito pertinere cenſeatur cujus beneficio curvarum areæ & longitudines &c. (id modo fiat) exacte & Geometrice determinentur. Sed iſta narrandi non eſt locus, reſpicienti duo præ reliquis demonſtranda occurrunt.

1. *Demonſtratio quadraturæ curvarum ſimplicium in Regula prima.*

Præparatio pro Regula prima demonſtranda.

Sit itaque curvæ alicujus $AD\delta$ Baſis $AB = x$, perpendiculariter applicata $BD = y$, & area $ABD = z$, ut prius. Item ſit $B\beta = o$, $BK = v$, & rectangulum $B\beta HK$ (ov) æquale ſpatio $B\beta\delta D$.



Eſt ergo $A\beta = x + o$, & $A\delta\beta = z + ov$. His præmiſſis, ex relatione inter x & z ad arbitrium aſſumpta quæro y iſto, quem ſequentem vides, modo.

Pro lubitu ſumatur $\frac{2}{3}x^{\frac{3}{2}} = z$, ſive $\frac{4}{9}x^3 = zz$. Tum $x + o$ $(A\beta)$ pro x, & $z + ov$ $(A\delta\beta)$ pro z ſubſtitutis, prodibit $\frac{4}{9}$ in $x^3 + 3x^2o + 3xo^2 + o^3 =$ (ex natura curvæ) $z^2 + 2zov + o^2v^2$.

Et

Et ſublatis ($\frac{4}{9}x^3$ & zz) æqualibus, reliquiſque per o diviſis, reſtat $\frac{4}{9}$ in $3x^2 + 3xo + o^2 = 2zv + ov^2$. Si jam ſupponamus $B\beta$ in infinitum diminui & evaneſcere, ſive o eſſe nihil, erunt v & y æquales, & termini per o multiplicati evaneſcent, quare reſtabit $\frac{4}{9} \times 3xx = 2zv$, ſive $\frac{2}{3}xx\ (= zy) = \frac{2}{3}x^{\frac{3}{2}}y$, ſive $x^{\frac{1}{2}}\ (= \frac{x^2}{x^{\frac{3}{2}}}) = y$. Quare e contra ſi $x^{\frac{1}{2}} = y$, erit $\frac{2}{3}x^{\frac{3}{2}} = z$.

Demonſtratio.

Vel generaliter, ſi $\frac{n}{m+n} \times ax^{\frac{m+n}{n}} = z$; ſive, ponendo $\frac{na}{m+n} = c$, & $m+n=p$, ſi $cx^{\frac{p}{n}} = z$, ſive $c^n x^p = z^n$: tum $x + o$ pro x, & $z + ov$ (ſive, quod perinde eſt, $z + oy$) pro z, ſubſtitutis, prodit c^n in $x^p + pox^{p-1}$, &c. $= z^n + noyz^{n-1}$, &c. reliquis nempe terminis, qui tandem evaneſcerent, omiſſis. Jam ſublatis $c^n x^p$ & z^n æqualibus, reliquiſque per o diviſis, reſtat $c^n px^{p-1} = nyz^{n-1}\ (= \frac{nyz^n}{z} = \frac{nyc^n x^p}{cx^{\frac{p}{n}}}$ ſive, dividendo per $c^n x^p$, erit $px^{-1} = \frac{ny}{cx^{\frac{p}{n}}}$ ſive $pcx^{\frac{p-n}{n}} = ny$; vel reſtituendo $\frac{na}{m+n}$ pro c, & $m+n$ pro p, hoc eſt, m pro $p-n$, & na pro pc, fiet $ax^{\frac{m}{n}} = y$. Quare e contra, ſi $ax^{\frac{m}{n}} = y$, erit $\frac{n}{m+n}ax^{\frac{m+n}{n}} = z$. Q. E. D.

Inventio curvarum quæ poſſunt quadrari.

Hinc in tranſitu notetur modus quo curvæ tot quot placuerit, quarum areæ ſunt cognitæ, poſſunt inveniri; ſumendo nempe quamlibet æquationem pro relatione inter aream z & baſin x ut inde quæratur applicata y. Ut ſi ſupponas $\sqrt{aa+xx} = z$, ex calculo invenies $\frac{x}{\sqrt{aa+xx}} = y$. Et ſic de reliquis.

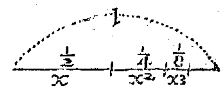
2. *Demonſtratio reſolutionis æquationum affectarum.*

Alterum demonſtrandum eſt literalis æquationum affectarum reſolutio. Nempe quod Quotiens, cum x ſit ſatis parva, quo magis producitur eo magis ad veritatem accedit, ut defectus (p, q, vel r, &c.) quo diſtat ab exacto valore

valore ipsius y, tandem evadat minor quavis data quantitate; & in infinitum producta fit ipsi y æqualis. Quod sic patebit.

1. Quoniam ex ultimo termino æquationum quarum p, q, r, &c. sunt radices, quantitas illa in qua x est minimæ dimensionis (hoc est, plusquam dimidium istius ultimi termini, si supponis x satis parvam esse) in qualibet operatione perpetuo tollitur: iste ultimus terminus (per 1. 10. *Elem.*) tandem evadet minor quavis data quantitate; & prorsus evanescet si opus infinite continuatur.

Nempe si $x = \frac{1}{2}$, erit x dimidium omnium $x + x^2 + x^3 + x^4$, &c. et x^2 dimidium omnium $x^2 + x^3 + x^4 + x^5$, &c. Itaque si $x \sqsupset \frac{1}{2}$, erit x plusquam dimidium omnium $x + x^2 + x^3$, &c. et x^2 plusquam dimidium omnium $x^2 + x^3 + x^4$, &c. Sic si $\frac{x}{b} \sqsupset \frac{1}{2}$, erit x plusquam dimidium omnium $x + \frac{x^2}{b} + \frac{x^3}{bb}$, &c. Et sic de reliquis. Et numeros coefficientes quod attinet, illi plerumque decrescunt perpetuo, vel si quando increscant, tantum opus est ut x aliquoties adhuc minor supponatur.



2. Si ultimus terminus alicujus æquationis continuo diminuatur donec tandem evanescat, una ex ejus radicibus etiam diminuetur donec cum ultimo termino simul evanescat.

3. Quare quantitatum p, q, r, &c. unus valor continuo decrescit donec tandem, cum opus in infinitum producitur, penitus evanescat.

4. Sed valores istarum p, q, vel r, &c. una cum quotiente eatenus extracta adæquant radices æquationis propositæ (Sic in resolutione æquationis $y^3 + aay + axy - 2a^3 - x^3 = 0$, supra ostensa, percipies $y = a + p = a - \frac{1}{4}x + q = a - \frac{1}{4}x + \frac{xx}{64a} + r$, &c.) Unde satis liquet propositum quod quotiens infinite producta est una ex valoribus de y.

Idem patebit substituendo quotientem pro y in æquationem propositam. Videbis enim terminos illos sese perpetuo destruere in quibus x est minimarum dimensionum.

CUM in Epistolis D. Newtoni, *vel in lucem jamdudum editis, vel quæ in manus nostras inciderunt, reperiantur aliqua quæ ad hanc Doctrinam pertinent, ea excerpere & huic Tractatui adjungere visum est.*

EXCERPTA

Ex Epiſtolis D. *NEWTONI* Ad Methodum FLUXIONUM, ET SERIERUM INFINITARUM Spectantibus.

Fragmentum **Epiſtolæ ad D.* Oldenburgium 13 Junii 1676 *miſſæ.*

FRactiones in Infinitas Series reducuntur per diviſionem ; & quantitates radicales per extractionem radicum, perinde inſtituendo operationes iſtas in ſpeciebus ac inſtitui ſolent in decimalibus numeris. Hæc ſunt fundamenta harum reductionum ; ſed extractiones radicum, multum abbreviantur per hoc *Theorema.*

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \tfrac{m}{n}AQ + \tfrac{m-n}{2n}BQ + \tfrac{m-2n}{3n}CQ + \tfrac{m-3n}{4n}DQ + \&c.$$

Ubi $P+PQ$ ſignificat quantitatem cujus Radix, vel etiam dimenſio quævis, vel radix dimenſionis, inveſtiganda eſt. P, primum terminum quantitatis ejus ; Q, reliquos terminos diviſos per primum. Et $\frac{m}{n}$, numeralem indicem dimenſionis ipſius $P+PQ$: Sive dimenſio illa integra ſit ; ſive (ut ita loquar) fracta ; ſive affirmativa, ſive negativa. Nam, ſicut Analyſtæ, pro *aa*, *aaa*, &c. ſcribere ſolent a^2, a^3, &c. ſic ego, pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c.a^5}$, &c. ſcribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$; & pro $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$, ſcribo a^{-1}, a^{-2}, a^{-3}.

Et

* *Extat Epiſtola in Tom. 3. Operum* Walliſii.

Et ſic pro $\frac{aa}{\sqrt{c:\overline{a^3+b^2x}}}$ ſcribo $aa\times\overline{a^3+b^2x}|^{-\frac{1}{3}}$; & pro $\frac{a^2b}{\sqrt{c:\overline{a^3+b^2x}\times\overline{a^3+b^2x}}}$. ſcribo $a^2b\times\overline{a^3+b^2x}|^{-\frac{2}{3}}$: In quo ultimo caſu, ſi $\overline{a^3+b^2x}|^{-\frac{2}{3}}$ concipiatur eſſe $\overline{P+PQ}|^{\frac{m}{n}}$ in Regula; erit $P=a^3$, $Q=\frac{b^2x}{a^3}$, $m=-2$, & $n=3$. Denique, pro terminis inter operandum inventis in quoto, uſurpo A, B, C, D, &c. nempe A pro primo termino $P^{\frac{m}{n}}$, B pro ſecundo $\frac{m}{n}AQ$, & ſic deinceps. Cæterum uſus Regulæ patebit exemplis.

Exempl. 1. Eſt $\sqrt{c^2+x^2}$ (ſeu $\overline{c^2+x^2}|^{\frac{1}{2}}$) $=c+\frac{x^2}{2c}-\frac{x^4}{8c^3}+\frac{x^6}{16c^5}-\frac{5x^8}{128c^7}+\frac{7x^{10}}{256c^9}$ &c. Nam, in hoc caſu, eſt $P=c^2$, $Q=\frac{x^2}{c^2}$, $m=1$, $n=2$, A $(=P^{\frac{m}{n}}=\overline{cc}|^{\frac{1}{2}})=c$, B$(=\frac{m}{n}AQ)=\frac{x^2}{2c}$, C $(=\frac{m-n}{2n}BQ)=-\frac{x^4}{8c^3}$, & ſic deinceps.

Exempl. 2. Eſt $\sqrt{5:\overline{c^5+c^4x-x^5}}$ (i. e. $\overline{c^5+c^4x-x^5}|^{\frac{1}{5}}$) $=c+\frac{c^4x-x^5}{5c^4}-\frac{2c^8x^2+4c^4x^6-2x^{10}}{25c^9}+$ &c. Ut patebit ſubſtituendo in allatam Regulam, 1 pro m, 5 pro n, c^5 pro P, & $\frac{c^4x-x^5}{c^5}$ pro Q. Poteſt etiam $-x^5$ ſubſtitui pro P, & $\frac{c^4x+c^5}{-x^5}$ pro Q, et tunc evadet $\sqrt{5:\overline{c^5+c^4x-x^5}}=-x+\frac{c^4x+c^5}{5x^4}+\frac{2c^8x^2+4c^9x+c^{10}}{25x^9}+$ &c. Prior modus eligendus eſt, ſi x valde parvum ſit; poſterior, ſi valde magnum.

Exempl. 3. Eſt $\frac{N}{\sqrt{3:\overline{y^3-a^2y}}}$ (hoc eſt, $N\times\overline{y^3-a^2y}|^{-\frac{1}{3}}$) æqualis $N\times\overline{\frac{1}{y}+\frac{a^2}{3y^3}+\frac{2a^4}{9y^5}+\frac{14a^6}{81y^7}+\text{\&c.}}$ Nam $P=y^3$, $Q=\frac{-aa}{yy}$, $m=-1$, $n=3$. A $(=P^{\frac{m}{n}}=y^{3\times-\frac{1}{3}})=y^{-1}$, hoc eſt $\frac{1}{y}$. B $(=\frac{m}{n}AQ=-\frac{1}{3}\times\frac{1}{y}\times\frac{-aa}{yy})$ $=\frac{aa}{3y^3}$, &c.

Exempl. 4. Radix cubica ex quadrato-quadrato ipſius $d+e$, (hoc eſt, $\overline{d+e}|^{\frac{4}{3}}$) eſt $d^{\frac{4}{3}}+\frac{4ed^{\frac{1}{3}}}{3}+\frac{2e^2}{9d^{\frac{2}{3}}}-\frac{4e^3}{81d^{\frac{5}{3}}}+$ &c.
Nam $P=d$, $Q=\frac{e}{d}$, $m=4$, $n=3$, A $(=P^{\frac{m}{n}})=d^{\frac{4}{3}}$, &c.

Exempl. 5. Eodem modo ſimplices etiam poteſtates eliciuntur. Ut ſi quadrato-cubus ipſius $d+e$, (hoc eſt, $\overline{d+e}|^5$, ſeu $\overline{d+e}|^{\frac{5}{1}}$) deſideretur: erit, juxta Regulam, $P=d$, $Q=\frac{e}{d}$, $m=5$, & $n=1$; adeoque A $(=P^{\frac{m}{n}})=d^5$, B $(=\frac{m}{n}AQ)=5d^4e$, & ſic $C=10d^3e^2$, $D=10d^2e^3$, $E=5de^4$, $F=e^5$, & G $(=\frac{m-5n}{6n}FQ)=0$. Hoc eſt, $\overline{d+e}|^5=d^5+5d^4e+10d^3e^2+10d^2e^3+5de^4+e^5$.

Exem.

Exempl. 6. Quinetiam Divisio, sive simplex sit, sive repetita, per eandem Regulam perficitur. Ut si $\frac{1}{d+e}$ (hoc est $\overline{d+e}|^{-1}$ sive $\overline{d+e}|^{\frac{-1}{1}}$) in seriem simplicium terminorum resolvendum sit: Erit juxta Regulam $P=d$, $Q=\frac{e}{d}$, $m=-1$, $n=1$, & $A\left(=P^{\frac{m}{n}}=d^{\frac{-1}{1}}\right)=d^{-1}$ seu $\frac{1}{d}$, $B\left(=\frac{m}{n}AQ=-1\times\frac{1}{d}\times\frac{e}{d}\right)=-\frac{e}{d^2}$, & sic $C=\frac{ee}{d^3}$, $D=-\frac{e^3}{d^4}$, &c. Hoc est $\frac{1}{d+e}=\frac{1}{d}-\frac{e}{d^2}+\frac{e^2}{d^3}-\frac{e^3}{d^4}+$ &c.

Exempl. 7. Sic et $\overline{d+e}|^{-3}$ (hoc est unitas ter divisa per $d+e$, vel semel per cubum ejus,) evadit $\frac{1}{d^3}-\frac{3e}{d^4}+\frac{6e^2}{d^5}-\frac{10e^3}{d^6}+$ &c.

Exempl. 8. Et $N\times\overline{d+e}|^{-\frac{1}{3}}$, (hoc est N divisum per radicem cubicam ipsius $d+e$,) evadit $N\times\frac{1}{d^{\frac{1}{3}}}-\frac{e}{3d^{\frac{4}{3}}}+\frac{2e^2}{9d^{\frac{7}{3}}}-\frac{14e^3}{81d^{\frac{10}{3}}}+$ &c.

Exempl. 9. Et $N\times\overline{d+e}|^{-\frac{3}{5}}$ (hoc est N divisum per radicem quadrato-cubicam ex cubo ipsius $d+e$, sive $\frac{N}{\sqrt{5: d^3+3d^2e+3de^2+e^3}}$) evadit $N\times\frac{1}{d^{\frac{3}{5}}}-\frac{3e}{5d^{\frac{8}{5}}}+\frac{12e^2}{25d^{\frac{13}{5}}}-\frac{52e^3}{125d^{\frac{18}{5}}}+$ &c.

Per eandem Regulam Genesis Potestatum, Divisiones per Potestates aut per quantitates radicales, & Extractiones radicum altiorum in numeris etiam commode instituuntur.

Extractiones Radicum affectarum in speciebus imitantur earum extractiones in numeris, sed methodus *Vietæ* & *Oughtredi* nostri huic negotio minus idonea est: Quapropter aliam excogitare adactus sum. [*Hujus specimen exhibetur in Tractatu præcedente Pag.* 8.]

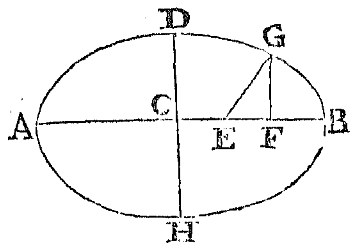
Quomodo ex æquationibus, sic ad infinitas series reductis, Areæ & Longitudines curvarum, Contenta & Superficies solidorum, vel quorumlibet segmentorum figurarum quarumvis, eorumque Centra gravitatis determinantur; & quomodo etiam Curvæ omnes Mechanicæ ad ejusmodi æquationes infinitarum serierum reduci possint, indeque Problemata circa illas resolvi perinde ac si Geometricæ essent; nimis longum foret describere, sufficiat specimina quædam talium Problematum recensuisse: Inque iis, brevitatis gratia, literas A, B, C, D, &c. pro terminis seriei, sicut sub initio, nonnunquam usurpabo.

1. *Si ex dato sinu recto, vel sinu verso, Arcus desideretur*: Sit radius r, & sinus rectus x: Eritque Arcus $=x+\frac{x^3}{6r^2}+\frac{3x^5}{40r^4}+\frac{5x^7}{112r^6}$ &c. hoc est, $=x+\frac{1\times1\times xx}{2\times3\times rr}A+\frac{3\times3xx}{4\times5rr}B+\frac{5\times5xx}{6\times7rr}C+\frac{7\times7xx}{8\times9rr}D+$ &c.

Vel, fit d diameter, ac x finus verfus ; & erit Arcus æqualis
$d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} +$ &c. hoc eft, $= \sqrt{dx}$ in
$1 + \frac{x}{6d} + \frac{3x^2}{40d^2} + \frac{5x^3}{112d^3} +$ &c.

2. Si viciffim, *ex dato Arcu defideretur finus* : Sit radius r, & arcus z : Eritque finus rectus $= z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \frac{z^9}{362880r^8} -$ &c. hoc eft, $= z - \frac{zz}{2 \times 3rr}A - \frac{zz}{4 \times 5rr}B - \frac{zz}{6 \times 7rr}C -$ &c. Et finus verfus $= \frac{z^2}{2r} - \frac{z^4}{24r^3} + \frac{z^6}{720r^5} - \frac{z^8}{40320r^7} +$ &c. hoc eft, $= \frac{zz}{1 \times 2r} - \frac{zz}{3 \times 4rr}A - \frac{zz}{5 \times 6rr}B - \frac{zz}{7 \times 8rr}C -$ &c.

3. *Si Arcus capiendus fit in ratione data ad alium Arcum* : Efto diameter $= d$, chorda arcus dati $= x$, & arcus quæfitus ad arcum illum datum ut n ad 1 ; Eritque arcus quæfiti Chorda $= nx + \frac{1-nn}{2 \times 3dd}xxA + \frac{9-nn}{4 \times 5dd}xxB + \frac{25-nn}{6 \times 7dd}xxC + \frac{36-nn}{8 \times 9dd}xxD + \frac{49-nn}{10 \times 11dd}xxE +$ &c. Ubi nota, quod cum n eft numerus impar, feries definet effe infinita, & evadet eadem quæ prodit per Vulgarem Algebram ad multiplicandum datum angulum per iftum numerum n.



4. Si in Axe alterutro AB, Ellipfeos ADB (cujus centrum C, & axis alter DH) detur punctum aliquod E, circa quod recta EG occurrens Ellipfi in G, motu angulari feretur ; & ex data area fectoris Elliptici BEG, quæratur recta GF, quæ a puncto G ad axem AB normaliter demittitur : Efto BC $= q$, DC $= r$, EB $= t$, ac duplum areæ BEG $= z$; et erit GF $= \frac{1}{t}z - \frac{q}{6r^2t^4}z^3 + \frac{10q^2 - 9qt}{120r^4t^7}z^5 - \frac{280q^3 + 504q^2t - 225qt^2}{5040r^6t^{10}}z^7 +$ &c. Sic itaque Aftronomicum illud *Kepleri* Problema refolvi poteft.

5. In eadem Ellipfi, fi ftatuatur CD $= r$, $\frac{CB^2}{CD} = c$, & CF $= x$: Erit arcus

$$\text{Ellipticus DG} = x + \frac{1}{6c^2}x^3 + \frac{1}{10rc^3}x^5 + \frac{1}{14r^2c^4}x^7 + \frac{1}{18r^3c^5}x^9 + \frac{1}{22r^4c^6}x^{11} + \text{\&c.}$$

$$\begin{array}{cccc}
-\frac{1}{4cc^4} & -\frac{1}{28rc^5} & -\frac{1}{24r^2c^6} & -\frac{1}{22r^3c^7} \\
 & +\frac{1}{112c^6} & +\frac{1}{48rc^7} & +\frac{3}{88r^2c^8} \\
 & & -\frac{5}{1152c^8} & -\frac{5}{352rc^9} \\
 & & & +\frac{7}{2816c^{10}}
\end{array}$$

Hic numerales Coefficientes fupremorum terminorum ($\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{14}$, &c.) funt in Mufica progreffione : Et numerales Coefficientes omnium inferiorum in unaquaque columna prodeunt multiplicando continuo numeralem Coefficientem fupremi termini per terminos hujus progreffionis

$\frac{\frac{1}{2}n-1}{2}$

$\frac{\frac{1}{2}n-1}{2}$, $\frac{\frac{3}{3}n-3}{4}$, $\frac{\frac{5}{4}n-5}{6}$, $\frac{\frac{7}{5}n-7}{8}$, $\frac{\frac{9}{6}n-9}{10}$, &c. Ubi n ſignificat numerum dimenſionum ipſius c in denominatore iſtius ſupremi termini. E. g. ut terminorum infra $\frac{1}{22r^4c^6}$, numerales coefficientes inveniantur, pono $n=6$, ducoque $\frac{1}{22}$ (numeralem coefficientem ipſius $\frac{1}{22r^4c^6}$) in $\frac{\frac{1}{2}n-1}{2}$, hoc eſt, in 1; & prodit $\frac{1}{22}$, numeralis coefficiens termini proxime inferioris: dein duco hunc $\frac{1}{22}$ in $\frac{\frac{3}{3}n-3}{4}$, ſive in $\frac{n-3}{4}$, hoc eſt, in $\frac{3}{4}$; & prodit $\frac{3}{88}$ numeralis coefficiens tertii termini in iſta columna. Atque iſta $\frac{3}{88}\times\frac{\frac{5}{4}n-5}{6}$ facit $\frac{5}{352}$ numeralem coefficientem quarti termini; & $\frac{5}{352}\times\frac{\frac{7}{5}n-7}{8}$ facit $\frac{7}{2816}$ numeralem coefficientem infimi termini. Idem in aliis ad infinitum uſque columnis præſtari poteſt: Adeoque valor ipſius DG per hanc Regulam pro lubitu produci.

Adhæc, ſi BF dicatur x, ſitque r latus rectum Ellipſeos, & $e=\frac{r}{AB}$; Erit Arcus Ellipticus

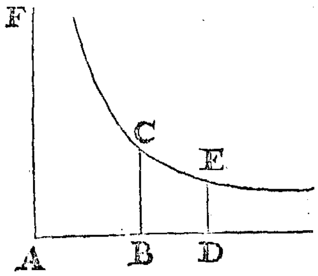
$$BG=\sqrt{rx}\ \text{in}\ 1+\left.\frac{2-\frac{3}{2}e}{3r}\right\}x\ \left.\frac{-2+3e-\frac{5}{8}e^2}{5r^2}\right\}x^2\ \left.\frac{+4-9e+\frac{23}{4}e^2-\frac{7}{16}e^3}{7r^3}\right\}x^3\ \left.\frac{-10+30e-\frac{123}{4}e^2+\frac{91}{8}e^3-\frac{45}{128}e^4}{9r^4}\right\}x^4,+\&c.$$

Quare, ſi ambitus totius Ellipſeos deſideretur; Biſeca CB in F, & quære Arcum DG, per prius Theorema, & Arcum BG per poſterius.

6. Si, vice verſa, *ex dato arcu Elliptico* DG, *quæratur Sinus ejus* CF; tum dicto CD $=r$, $\frac{CB^2}{CD}=c$, & arcu illo DG $=z$; Erit

$$CF=z-\frac{1}{6c^2}z^3-\left(\frac{1}{10rc^3}+\frac{13}{120c^4}\right)z^5-\left(\frac{1}{14r^2c^4}+\frac{71}{420rc^5}-\frac{493}{5040c^6}\right)z^7-\&c.$$

Quæ autem de Ellipſi dicta ſunt, omnia facile accommodantur ad Hyperbolam; mutatis tantum ſignis ipſorum c & e ubi ſunt imparium dimenſionum.

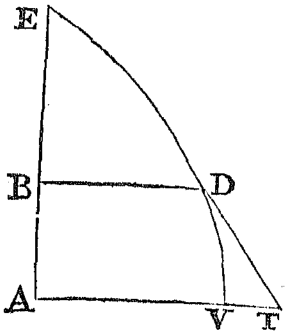


7. Præterea, ſi ſit CE Hyperbola, cujus Aſymptoti AD, AF rectum angulum FAD conſtituant; & ad AD erigantur utcunque perpendicula BC, DE occurrentia Hyperbolæ in C & E: & AB dicatur a, BC b, & area BCED z;

Erit

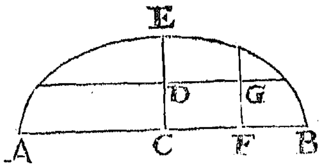
Erit BD $= \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} +$ &c. Ubi coefficientes denominatorum prodeunt multiplicando terminos hujus Arithmeticæ progreſſionis, 1, 2, 3, 4, 5, &c. in ſe continuo. Et hinc ex Logarithmo dato poteſt numerus ei competens invenire.

8. Eſto VDE *Quadratrix*, cujus vertex eſt V, exiſtente A centro & AE ſemi-diametro Circuli ad quem aptatur, & angulo VAE recto: Demiſſoque ad AE perpendiculo quovis DB, & acta Quadratricis Tangente DT occurrente axi ejus AV in T: Dic AV $= a$, & AB $= x$; Eritq; DB $= a - \frac{x^2}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5} -$ &c. Et VT $= \frac{x^2}{3a} + \frac{x^4}{15a^3} + \frac{2x^6}{189a^5} +$ &c. Et Area AVDB $= ax - \frac{x^3}{9a} - \frac{x^5}{225a^3} - \frac{2x^7}{6615a^5} -$ &c. Et Arcus VD $= x + \frac{2x^3}{27a^2} + \frac{14x^5}{2025a^4} + \frac{604x^7}{893025a^6} +$ &c. Unde viciſſim, ex dato BD, vel VT, aut area AVDB, arcuve VD, per reſolutionem affectarum æquationum erui poteſt x ſeu AB.



9. Eſto denique AEB *Sphæroides*, revolutione Ellipſeos AEB circa axem AB genita, & ſecta planis quatuor, AB per axem tranſeunte, DG parallelo AB, CDE perpendiculariter biſecante axem, & FG parallelo CE: ſitque recta CB $= a$, CE $= c$, CF $= x$, & FG $= y$. Et Sphæroideos ſegmentum CDGF dictis quatuor planis comprehenſum, erit

$$\begin{aligned}
&+ 2cxy - \frac{x}{3c}y^3 - \frac{x}{20c^3}y^5 - \frac{x}{56c^5}y^7 - \frac{5x}{576c^7}y^9 - \text{\&c.} \\
&- \frac{cx^3}{3a^2} - \frac{x^3}{18ca^2} - \frac{x^3}{40c^3a^2} - \frac{5x^3}{336c^5a^2} - \text{\&c.} \\
&- \frac{cx^5}{20a^4} - \frac{x^5}{40ca^4} - \frac{3x^5}{160c^3a^4} - \text{\&c.} \\
&- \frac{cx^7}{56a^6} - \frac{5x^7}{336ca^6} - \text{\&c.} \\
&- \frac{5cx^9}{576a^8} - \text{\&c.} \\
&- \text{\&c.}
\end{aligned}$$



Ubi numerales coefficientes ſupremorum terminorum ($2, -\frac{1}{3}, -\frac{1}{20}, -\frac{1}{56}, -\frac{5}{576}$, &c.) in infinitum producuntur multiplicando primum coefficientem 2 continuo per terminos hujus progreſſionis $-\frac{1\times1}{2\times3}, \frac{1\times3}{4\times5}, \frac{3\times5}{6\times7}, \frac{5\times7}{8\times9}, \frac{7\times9}{10\times11}$, &c. Et numerales coefficientes terminorum in unaquaque columna deſcendentium in infinitum producuntur multiplicando continuo coefficientem ſupremi termini in prima columna per eandem progreſſionem, in ſecunda autem per terminos hujus $\frac{1\times1}{2\times3}, \frac{3\times3}{4\times5}, \frac{5\times5}{6\times7}, \frac{7\times7}{8\times9}$, &c. in tertia per terminos hujus $\frac{3\times1}{2\times3}, \frac{5\times3}{4\times5}, \frac{7\times5}{6\times7}, \frac{9\times7}{8\times9}$, &c. in quarta per terminos hujus $\frac{5\times1}{2\times3}, \frac{7\times3}{4\times5}, \frac{9\times5}{6\times7}$, &c. in quinta per terminos hujus $\frac{7\times1}{2\times3}, \frac{9\times3}{4\times5}, \frac{11\times5}{6\times7}$, &c. Et ſic in infinitum.

Et

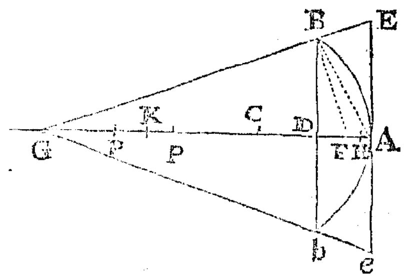
Et eodem modo ſegmenta aliorum ſolidorum deſignari, & valores eorum aliquando commode per ſeries quaſdem numerales in infinitum produci poſſunt.

Ex his videre eſt, quantum fines Analyſeos per hujuſmodi infinitas æquationes amplientur: Quippe quæ, earum beneficio, ad omnia pæne dixerim problemata (ſi numeralia *Diophanti* & ſimilia excipias) ſeſe extendit.

Non tamen omnino univerſalis evadit, niſi per ulteriores quaſdam methodos eliciendi ſeries infinitas. Sunt enim quædam Problemata in quibus non liceat ad ſeries infinitas per diviſionem vel extractionem radicum ſimplicium affectarumve, pervenire. Sed quomodo in iſtis caſibus procedendum ſit, jam non vacat dicere; ut neque alia quædam tradere quæ circa reductionem infinitarum ſerierum in finitas, ubi rei natura tulerit, excogitavi. Nam parcius ſcribo, quod hæ ſpeculationes diu mihi faſtidio eſſe cœperint, adeo ut ab iiſdem jam per quinque fere annos abſtinuerim.

Unum tamen addam: quod poſtquam Problema aliquod ad infinitam æquationem deducitur, poſſint inde variæ approximationes in uſum Mechanicæ, nullo fere negotio formari; quæ, per alias methodos quæſitæ, multo labore temporiſque diſpendio conſtare ſolent.

Cujus rei exemplo eſſe poſſunt Tractatus *Hugenii* aliorumque de Quadratura circuli. Nam, ut ex data arcus chorda A, & dimidii arcus chorda B, arcum illum proxime aſſequaris; finge arcum illum eſſe z, & circuli radium r; juxtaque ſuperiora erit A (nempe duplum ſinus dimidii z) $= z - \frac{z^3}{4 \times 6r^2} + \frac{z^5}{4 \times 4 \times 120r^4} -$ &c. Et $B = \frac{1}{2}z - \frac{z^3}{2 \times 16 \times 6r^2} + \frac{z^5}{2 \times 16 \times 16 \times 120r^4} -$ &c. Duc jam B in numerum fictitium n, & a producto aufer A, & reſidui ſecundum terminum (nempe $- \frac{nz^3}{2 \times 16 \times 6r^2} + \frac{z^3}{4 \times 6r^2}$,) eo ut evaneſcat, pone $= 0$; indeque emerget $n = 8$, et erit $8B - A = 3z * - \frac{3z^5}{64 \times 120r^4} +$ &c. hoc eſt $\frac{8B - A}{3} = z$; errore tantum exiſtente $\frac{z^5}{7680r^4} -$ &c. in exceſſu. Quod eſt Theorema *Hugenianum.*



Inſuper, ſi in Arcus Bb, ſagitta AD indefinite producta, quæratur punctum G, a quo actæ rectæ GB, Gb abſcindant Tangentem Ee quam proxime æqualem Arcui iſti: Eſto circuli centrum C, diameter $AK = d$, & ſagitta $AD = x$: Et erit DB $(= \sqrt{dx - x^2})$

$$= d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16d^{\frac{5}{2}}} - \text{\&c.}$$

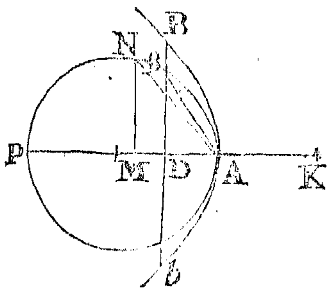
Et AE $(= AB) = d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} +$ &c.

Et AE — DB : AD : : AE : AG ; Quare $AG = \frac{3}{2}d - \frac{1}{5}x - \frac{12x^2}{175d}$ — vel + &c. Finge ergo $AG = \frac{3}{2}d - \frac{1}{5}x$; et viciſſim erit DG $(\frac{3}{2}d - \frac{6}{5}x)$: DB : : DA : AE — DB. Quare $AE - DB = \frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} + \frac{23x^{\frac{7}{2}}}{300d^{\frac{5}{2}}}$ + &c. Adde DB ; & prodit

$AE = d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{17x^{\frac{7}{2}}}{1200d^{\frac{5}{2}}}$ + &c. Hoc aufer de valore ipſius AE ſupra habito, & reſtabit error $\frac{16x^{\frac{7}{2}}}{525d^{\frac{5}{2}}}$ + vel — &c. Quare in AG, cape AH quintam partem DA, et KG = HC, et actæ GBE, G*be* abſcindent Tangentem E*e* quam proxime æqualem arcui BA*b* ; errore tantum exiſtente $\frac{16x^3}{525d^3}\sqrt{dx}$ + vel — &c. multo minore ſcilicet quam in Theoremate *Hugenii.* Quod ſi fiat 7AK : 3AH : : DH : *n* ; et capiatur KG = CH — *n*, erit error adhuc multo minor.

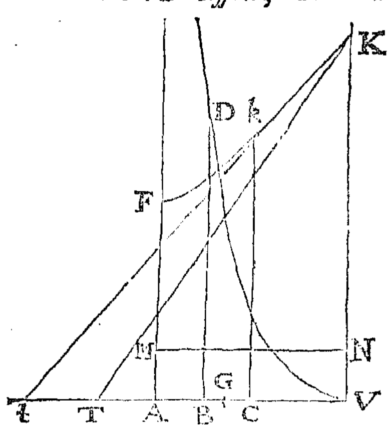
Atque ita, ſi Circuli ſegmentum aliquod BA*b* per Mechanicam deſignandum eſſet : Primo reducerem Aream iſtam in Infinitam ſeriem, puta hanc $BbA = \frac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{36d^{\frac{5}{2}}}$ — &c. Dein quærerem conſtructiones Mechanicas quibus hanc ſeriem proxime aſſequerer ; cujuſmodi ſunt hæ : Age rectam AB, et erit ſegmentum $BbA = \overline{\tfrac{2}{3}AB + BD} \times \tfrac{4}{5}AD$ proxime ; exiſtente ſcilicet errore tantum $\frac{x^3}{70d^2}\sqrt{dx}$ + &c. in defectu : Vel proximius, erit ſegmentum illud (biſecto AD in F, & acta recta BF) $= \frac{4BF + AB}{15} \times 4AD$; exiſtente errore ſolummodo $\frac{x^3}{560d^2}\sqrt{dx}$ + &c. qui ſemper minor erit quam $\frac{1}{1500}$ totius ſegmenti, etiamſi ſegmentum illud ad uſque ſemicirculum augeatur.

Sic et in Ellipſi BA*b*, [*Vid. Fig. Præcedent.*] cujus vertex A, axis alteruter AK, & latus rectum AP ; cape $PG = \frac{1}{2}AP + \frac{19AK - 21AP}{10AK} \times AP$. In Hyperbola vero, cape $PG = \frac{1}{2}AP + \frac{19AK + 21AP}{10AK} \times AP$. Et acta recta GBE abſcindet tangentem AE quam proxime æqualem arcui Elliptico vel Hyperbolico AB, dummodo Arcus ille non ſit nimis magnus.



Et pro Area ſegmenti Hyperbolici B*b*A ; in DP cape $MD = \frac{3AD^2}{4AK}$, & ad D & M erige perpendicula Dβ, MN occurrentia ſemicirculo ſuper Diametro AP deſcripto : Eritque $\frac{4AN + A\beta}{15} \times 4AD = BbA$ proxime : Vel proximius, erit $\frac{21AN + 4A\beta}{75} \times 4AD = BbA$; ſi modo capiatur $DM = \frac{5AD^2}{7AK}$.

Fragmentum Epistolæ D. Newtoni, *ad* D. Oldenburgium 24 Octob. 1676 *missæ*.



LOngitudo *Cissoidis* sic construitur. Sit VD *Cissois*, AV Diameter Circuli ad quem aptatur, V vertex, AF Asymptota ejus, ac DB perpendiculare quodvis ad AV demissum. Cum semi-axe AF = AV, & semi-parametro AG = $\frac{1}{3}$AV, describatur Hyperbola FkK; & inter AB & AV sumpta AC media proportionali, erigantur ad C & V perpendicula Ck, VK Hyperbolæ occurrentia in k & K; Et agantur rectæ KT, kt tangentes Hyperbolam in eisdem K & k, et occurrentes AV in T & t; Et ad AV constituatur rectangulum AVNM æquale spatio TKkt. Et *Cissoidis* VD longitudo erit Sextupla altitudinis VN. Demonstratio perbrevis est.

[*Quæ sequuntur scripta sunt in explicationem Epistolæ præcedentis.*]

Quod vero attinet ad Inventionem terminorum p, q, r, (*vide* pag. 25 & 8 :) in extractione Radicis affectæ, primum p sic eruo.

Descripto Angulo recto BAC, latera ejus BA, CA divido in partes æquales; & inde normales erigo distribuentes angulare spatium in æqualia parallelograma vel quadrata, quæ concipio denominata esse a dimensionibus duarum indefinitarum specierum, puta x & y, regulariter ascendentium a termino A; prout vides in Fig. 1. inscriptas. Ubi y denotat Radicem extrahendam; et x alteram indefinitam quantitatem, ex cujus potestatibus series conficienda;

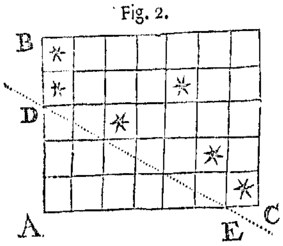
B Fig. 1.

x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
o	y	y^2	y^3	y^4

A C

enda eſt. Deinde, cum Æquatio aliqua proponitur, parallelogramma ſingulis ejus terminis correſpondentia inſignio nota aliqua: Et Regula ad duo vel forte plura ex inſignitis parallelogrammis applicata; quorum unum ſit humillimum in columna ſiniſtra juxta AB, & alia ad Regulam dextrorſum ſita, cæteraque omnia non contingentia Regulam ſupra eam jaceant; Seligo terminos Æquationis per parallelogramma contingentia Regulam deſignatos, & inde quæro quantitatem Quotienti addendam.

Fig. 2.



Sic ad extrahendam Radicem y, ex $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$; parallelogramma hujus terminis reſpondentia ſigno nota aliqua $\ast$; ut vides in Fig. 2. Dein applico Regulam DE ad inferiorem e locis ſignatis in ſiniſtra columna; eamque ab inferioribus ad ſuperiora dextrorſum gyrare facio, donec alium ſimiliter vel forte plura e reliquis ſignatis locis cœperit attingere. Videoque loca ſic attracta eſſe x^3, x^2y^2, & y^6. E terminis itaque, $y^6 - 7a^2x^2y^2 + 6a^3x^6$ tanquam nihilo æqualibus (& inſuper ſi placet reductis ad $v^6 - 7v^2 + 6 = 0$, ponendo $y = v\sqrt{ax}$,) quæro valorem y, & invenio quadruplicem, $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, & $-\sqrt{2ax}$, quorum quemlibet pro primo termino Quotientis accipere licet, prout e radicibus quampiam extrahere decretum eſt.

Sic Æquatio $y^3 + axy + aay - x^3 - 2a^3 = 0$, quam reſolvebam in priori Epiſtola, dat $-2a^3 + aay + y^3 = 0$, & inde $y = a$ proxime: Cum itaque a ſit primus terminus valoris y, pono p pro cæteris omnibus in infinitum, & ſubſtituo $a + p = y$. (Obvenient hic aliquando difficultates nonnullæ; ſed ex iis, credo, lector ſe proprio marte extricabit.) Subſequentes vero termini q, r, s, &c. eodem modo ex æquationibus ſecundis, tertiis, cæteriſque eruuntur, quo primus p e prima, ſed cura leviori; quia cæteri valores y ſolent prodire dividendo terminum involventem infimam poteſtatem indefinitæ quantitatis x per Coefficientem radicis p, q, r, aut s.

Intellexti credo ex ſuperioribus, regreſſionem ab Areis curvarum ad Lineas rectas, fieri per hanc extractionem Radicis affectæ. Sed duo alii ſunt modi quibus idem perficio.

Eorum unus affinis eſt Computationibus quibus colligebam approximationes ſub finem alterius Epiſtolæ, & intelligi poteſt per hoc exemplum. Proponatur Æquatio ad Aream Hyperbolæ $z = x + \frac{1}{2}xx + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5$, &c. Et partibus ejus multiplicatis in ſe, emerget $z^2 = x^2 + x^3 + \frac{11}{12}x^4 + \frac{5}{6}x^5$, &c. $z^3 = x^3 + \frac{3}{2}x^4 + \frac{7}{4}x^5$, &c. $z^4 = x^4 + 2x^5$, &c. $z^5 = x^5$, &c. Jam de z aufero $\frac{1}{2}zz$, & reſtat $z - \frac{1}{2}zz = x - \frac{1}{6}x^3 - \frac{5}{24}x^4 - \frac{13}{60}x^5$, &c. Huic addo $\frac{1}{6}z^3$, & fit $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 = x + \frac{1}{24}x^4 + \frac{3}{40}x^5$, &c. Aufero $\frac{1}{24}z^4$, & reſtat $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 = x - \frac{1}{120}x^5$, &c. Addo $\frac{1}{120}z^5$, & fit $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 + \frac{1}{120}z^5 = x$ quamproxime; ſive $x = z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c.

Eodem

Eodem modo, Series de una indefinita quantitate, in aliam transferri poſſunt. Quemadmodum ſi poſito r radio circuli, x ſinu recto arcus z, & $x + \frac{x^3}{6rr} + \frac{3x^5}{40r^4} +$ &c. longitudine arcus iſtius; atque hanc Seriem a Sinu recto ad Tangentem vellem transferre: Quæro longitudinem Tangentis $\frac{rx}{\sqrt{rr-xx}}$, & reduco in infinitam Seriem $x + \frac{x^3}{2rr} + \frac{3x^5}{8r^4} +$ &c. Vocetur hæc quantitas, t. Colligo poteſtates ejus $t^3 = x^3 + \frac{3x^5}{2rr}$ &c. $t^5 = x^5 +$ &c. Aufero autem t de z, & reſtat $z - t = -\frac{1}{3}x^3 - \frac{3}{10}x^5 -$ &c. Addo $\frac{1}{3}t^3$, & fit $z - t + \frac{1}{3}t^3 = \frac{1}{5}x^5 +$ &c. Aufero $\frac{1}{5}t^5$, & reſtat $z - t + \frac{1}{3}t^3 - \frac{1}{5}t^5 = 0$ quamproxime. Quare eſt $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 -$ &c. Sed ſiquis in uſus Trigonometricos me juſſiſſet exhibere expreſſionem Arcus per Tangentem; eam non hoc circuitu, ſed directa methodo quæſiviſſem.

Per hoc genus Computi colliguntur etiam Series ex duabus vel pluribus indefinitis quantitatibus conſtantes; & Radices affectarum Æquationum magna ex parte extrahuntur. Sed ad hunc poſteriorem uſum adhibeo potius methodum in altera Epiſtola deſcriptam tanquam generaliorem, & (Regulis pro Eliſione ſuperfluorum terminorum habitis) paulo magis expeditam.

Pro Regreſſione vero ab Areis ad Lineas rectas, & ſimilibus, poſſunt hujuſmodi *Theoremata* adhiberi.

THEOREMA I. Sit $z = ay + by^2 + cy^3 + dy^4 + ey^5$, &c.

Et viciſſim erit
$$y = \frac{z}{a} - \frac{b}{a^3}z^2 + \frac{2b^2 - ac}{a^5}z^3 + \frac{5abc - 5b^3 - a^2d}{a^7}z^4 + \frac{3a^2c^2 - 21ab^2c + 6a^2bd + 14b^4 - a^3e}{a^9}z^5 + \text{\&c.}$$

Exempli gratia. Proponatur Æquatio ad Aream Hyperbolæ, $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$ &c. Et ſubſtitutis in Regula 1 pro a, $-\frac{1}{2}$ pro b, $\frac{1}{3}$ pro c, $-\frac{1}{4}$ pro d, & $\frac{1}{5}$ pro e; viciſſim exurgit, $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 +$ &c.

THEOREMA II. Sit $z = ay + by^3 + cy^5 + dy^7 + ey^9 +$ &c.

Et viciſſim erit
$$y = \frac{z}{a} - \frac{b}{a^4}z^3 + \frac{3b^2 - ac}{a^7}z^5 + \frac{8abc - a^2d - 12b^3}{a^{10}}z^7 + \frac{55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - a^3e}{a^{13}}z^9 + \text{\&c.}$$

Exempli gratia. Proponatur Æquatio ad Arcum circuli, $z = y + \frac{y^3}{6rr} + \frac{3y^5}{40r^4} + \frac{5y^7}{112r^6} +$ &c. Et ſubſtitutis in Regula 1 pro a, $\frac{1}{6r^2}$ pro b, $\frac{3}{40r^4}$ pro c, $\frac{5}{112r^6}$ pro d &c; orietur $y = z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} +$ &c.

Fragmentum **Epiſtolæ* D. Newtoni, *ad* D. Wallisium *Anno* 1692, *miſſæ.*

SUB finem Epiſtolæ anni 1676 [*Hæc ſunt verba* Walliſii] ſcribit [D. *Newtonus*] etiam Problema determinandi Curvas per conditiones Tangentium in ſua poteſtate eſſe, una cum aliis difficilioribus; ad quæ ſolvenda ſe uſum eſſe dicit duplici methodo, una concinniore, altera generaliore; & utramque literis tranſpoſitis celat: quæ in ordinem redactæ hanc ſententiam exhibent. *Una methodus conſiſtit in Extractione fluentis quantitatis ex æquatione ſimul involvente fluxionem ejus. Altera tantum in aſſumptione ſeriei pro quantitate qualibet incognita ex qua cætera commode derivari poſſunt; & in collatione terminorum homologorum æquationis reſultantis ad eruendos terminos aſſumptæ Seriei.* Harum methodorum Secunda ex verbis jam recitatis abſque ulteriore explicatione intelligi poteſt; priorem ab Authore jam accepi ut ſequitur.

Hæc methodus, ait, ejuſdem eſt generis cum ea pro extrahendo radices ex æquationibus affectis ſuperius deſcripta. Pone quod Problema reſolvendum reducatur ad æquationem fluentes quantitates y & z una cum earum fluxionibus $\dot{y}$ & $\dot{z}$ involventem, & quod fluxio ipſius z uniformis ſit. Ut hæc fluxio ex æquatione evaneſcat, pro ea ponatur unitas, & manebit æquatio ſolas y, z & $\dot{y}$ involvens, *quam Reſolvendam* vocat. Proponitur, inventio ipſius y in Serie infinita convergente, quæ ſolam z involvet. Hoc in aliquibus æquationibus impoſſibile eſt, in aliis præparationem æquationum requirit, ubi vero directe confici poſſit reſolutio eſt hujuſmodi.

PROBLEMA

* *Extat Epiſtola in Tom. 2. Operum* Walliſii.

PROBLEMA.

Ex æquatione fluxionem radicis involvente radicem extrahere.

RESOLUTIO.

TErmini omnes, ex eodem æquationis latere consistentes, æquentur nihilo, & ipsarum y & $\dot{y}$ dignitates (si opus sit) exaltentur vel deprimantur, sic ut earum indices nec alicubi negativi sint, nec tamen altiores quam ad hunc effectum requiritur; & sit kz^{λ} terminus infimæ dignitatis eorum qui neque per y neque per ejus fluxionem $\dot{y}$ neque per earum dignitatem quamvis multiplicantur. Sit $lz^{\mu}y^{\alpha}\dot{y}^{\beta}$ terminus alius quilibet, & omnes ordine terminos percurrendo collige ex singulis seorsim numerum $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ sic, ut tot habeas ejusmodi numeros quot sunt termini. Horum numerorum maximus vocetur ν, & z^{ν} erit dignitas primi termini Seriei. Pro ejus coefficiente ponatur a, & in æquatione quæ *resolvenda* dicitur scribe az^{ν} pro y, & $\nu az^{\nu-1}$ pro $\dot{y}$; ac termini omnes resultantes in quibus z ejusdem est dignitatis ac in termino kz^{λ}, sub propriis signis collecti, ponantur æquales nihilo. Nam hæc æquatio debite reducta dabit coefficientem a. Sic habes az^{ν} terminum primum Seriei.

OPERATIO SECUNDA.

Pro reliquis omnibus hujus Seriei terminis nondum inventis pone p, & habebis Æquationem $y=az^{\nu}+p$, & inde etiam Æquationem $\dot{y}=\nu az^{\nu-1}+\dot{p}$. In resolvenda, pro y & $\dot{y}$ scribe hos eorum valores & habebis Resolvendam novam, ubi p officium præstat ipsius y: & ex hac Resolvenda primum extrahes terminum Seriei p eodem modo atque terminum primum Seriei totius $y=az^{\nu}+p$ ex Resolvenda prima extraxisti.

OPERATIO TERTIA ET SEQUENTES.

Dein tertiam Resolvendam eadem ratione invenias atque secundam invenisti, & ex ea terminum tertium Seriei totius extrahes. Et similiter Resolvendam quartam invenies, & ex ea quartum Seriei terminum, & sic in infinitum. Series autem sic inventa erit radix Æquationis quam extrahere oportuit.

EXEMPLUM.

EXEMPLUM.

Ex Æquatione $y^2\dot{z}^2 - z^2\dot{z}\dot{y} - d^2\dot{z}\dot{z} + dz\dot{z}^2 = 0$, extrahenda ſit radix y. Pone $\dot{z} = 1$, & Æquatio evadet $y^2 - z^2\dot{y} - dd + dz = 0$, quæ eſt Reſolvenda. Jam vero terminus infimus in quo nec y neque $\dot{y}$ reperitur, eſt dd, qui ipſi kz^λ æquatus dat $\lambda = 0$. Terminis reliquis y^2, $-z^2\dot{y}$ pone $lz^\mu y^\alpha \dot{y}^\beta$ æqualem ſucceſſive, & inde in primo caſu habebis $\mu = 0$, $\alpha = 2$, $\beta = 0$, in ſecundo $\mu = 2$, $\alpha = 0$, & $\beta = 1$. Et hinc $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ fit in primo caſu 0, in ſecundo -1. Unde ν eſt 0, & az^ν & $\nu az^{\nu - 1}$ ſunt a & 0; quarum ultimæ duæ a & 0 in Reſolvenda pro y & $\dot{y}$ ſcriptæ, producunt $aa + 0z^2 - dd + dz$; & termini aa & $-dd$, in quibus index dignitatis z eſt λ ſeu 0, poſiti æquales nihilo dant $a = d$. Unde primus Seriei terminus az^ν evadit d.

OPERATIO SECUNDA.

Pro terminis reliquis pone p, & habebis æquationem $y = d + p$, & inde $\dot{y} = \dot{p}$; qui valores in Reſolvenda pro y & $\dot{y}$ ſubſtituti dant Reſolvendam novam $2d\dot{p} + pp - zz\dot{p} + dz = 0$, ubi p & $\dot{p}$ vices ſubeunt ipſarum y & $\dot{y}$. Terminus unicus in quo nec p neque $\dot{p}$ reperitur eſt dz, qui cum termino kz^λ collatus dat $\lambda = 1$. Terminis reliquis $2d\dot{p}$, pp & $-zz\dot{p}$ pone $lz^\mu p^\alpha \dot{p}^\beta$ æqualem ſucceſſive; & inde in primo caſu habebis $\mu = 0$, $\alpha = 1$, & $\beta = 0$; in ſecundo $\mu = 0$, $\alpha = 2$, & $\beta = 0$; & in tertio $\mu = 2$, $\alpha = 0$, & $\beta = 1$. Et hinc $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ evadit primo caſu 1, in ſecundo $\frac{1}{2}$, in tertio 0. Unde ν eſt 1, & az^ν & $\nu az^{\nu - 1}$ ſunt az & a. Termini duo ultimi az & a in Reſolvenda pro p & $\dot{p}$ reſpective ſcripti, producunt $2daz + a^2z^2 - az^2 + dz$. Et termini $2daz$ & dz in quibus index dignitatis z eſt λ ſeu 1, poſiti æquales nihilo, dant $a = -\frac{1}{2}$. Unde az, terminus primus Seriei p fit $-\frac{1}{2}z$.

OPERATIO TERTIA.

Pro terminis reliquis nondum inventis pone q & habebis æquationem $p = -\frac{1}{2}z + q$, & inde $\dot{p} = -\frac{1}{2} + \dot{q}$: Qui valores pro p & $\dot{p}$ in Reſolvenda noviſſima ſubſtituti producunt Reſolvendam novam $2d\dot{q} - zq + qq + \frac{3}{4}zz - zz\dot{q} = 0$. Ubi q & $\dot{q}$ vices ſupplent ipſorum y & $\dot{y}$. Terminus unicus in quo neque q nec $\dot{q}$ reperitur eſt $\frac{3}{4}zz$, qui cum kz^λ collatus dat $\lambda = 2$. Terminis reliquis $2d\dot{q}$, $-zq$, $+qq$, $-zz\dot{q}$ pone $lz^\mu q^\alpha \dot{q}^\beta$ æqualem ſucceſſive; & inde in primo caſu habebis $\mu = 0$, $\alpha = 1$, & $\beta = 0$; in ſecundo,

$\mu = 1$,

$\mu = 1$, $\alpha = 1$, $\beta = 0$; in tertio, $\mu = 0$, $\alpha = 2$, $\beta = 0$: in quarto $\mu = 2$, $\alpha = 0$, $\beta = 1$: & inde $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ evadit in primo casu 2, in secundo, tertio, & quarto 1. Et hinc v est 2, vel az^v & vaz^{v-1} sunt az^2 & $2az$: qui valores in Resolvenda pro q & $\dot{q}$ substituti dant $2dazz - az^3 + aaz^4 + \frac{3}{4}zz - 2az^3$; & termini $2dazz + \frac{3}{4}zz$ in quibus index dignitatis z est λ seu 2, positi æquales nihilo, dant $a = -\frac{3}{8d}$. Unde az^v terminus primus Seriei q evadit $-\frac{3zz}{8d}$.

OPERATIO QUARTA.

Pro reliquis Seriei terminis nondum inventis pone r, & habebis æquationes $q = -\frac{3zz}{8d} + r$, & $\dot{q} = -\frac{3z}{4d} + \dot{r}$; & inde resolvendam novam $2dr + \frac{9z^3}{8d} - zr + \frac{9z^4}{64dd} - \frac{3zzr}{4d} + rr - zz\dot{r} = 0$; & ex ea per Methodum superiorem habebis $-\frac{9z^3}{16dd}$ terminum primum Seriei r. Et sic pergitur in infinitum.

Est igitur radix extrahenda $y = d + p = d - \frac{1}{2}z + q = d - \frac{1}{2}z - \frac{3zz}{8d} + r = d - \frac{1}{2}z - \frac{3zz}{8d} - \frac{9z^3}{16dd} - \&c.$ Et operationem continuando producere licet radicem ad terminos plures.

Et eadem methodo, dicit *Newtonus*, radices æquationum, fluxiones secundas, tertias, quartas, ($\ddot{y}, \dddot{y}, \ddddot{y}$,) aliasque involventium, extrahi posse.

His utitur radicum extractionibus ubi aliæ Methodi nil prosunt. Nam in Epistola prædicta anni 1676 docet, quod in Solutione problematum de Tangentibus inversorum, casus aliqui dantur in quibus hæc Methodus generalis non requiritur: & particulariter, si in triangulo rectangulo quod ab ordinata, tangente, & interjacente parte abscissæ constituitur, relatio duorum quorumlibet e lateribus tribus per æquationem quamvis definiatur; Problema absque Methodo hacce generali solvi poterit.

Methodi autem hæ omnes tam particulares quam generales collectim sumptæ, solutionem exhibent secundæ partis problematis, quod *Newtonus* sub initio istius Epistolæ his verbis proposuit. *Data æquatione quotcunque fluentes quantitates involvente fluxiones invenire, & vice versa.* Nam tota fluxionum Methodus in hujus directa & inversa solutione consistit.

Part of a Letter from Sir *Is. Newton*, to Mr. *J. Collins*, *Novemb.* 8. 1676.

THere is no Curve-line express'd by any Equation of three terms, tho' the unknown quantities affect one another in it, or the Indices of their Dignities be surd quantities (suppose $ax^{\lambda} + bx^{\mu}y^{\sigma} + cy^{\tau} = 0$, *where* x *signifies the Base,* y *the Ordinate,* λ, μ, σ, τ *the Indices of the dignities of* x *and* y, *and* a, b, c *known quantities with their signs* + *or* —, *) I say, there is no such Curve-line, but I can, in less than half a quarter of an hour, tell whether it may be Squar'd, or what are the simplest Figures it may be compar'd with, be those Figures Conic Sections, or others: And then by a direct and short way, (I dare say the shortest the nature of the thing admits of, for a general one,) I can compare them. And so if any two Figures express'd by such Equations be propounded, I can, by the same Rule compare them if they may be compar'd. This may seem a bold assertion, because it's hard to say a Figure may, or may not, be Squar'd, or Compar'd with another; but it's plain to me by the fountain I draw it from, tho' I will not undertake to prove it to others. The same Method extends to Equations of four Terms, and others also, but not so generally.*

Fragmentum Epistolæ D. Newtoni *ad D.* Collinsium, Novemb. 8. 1676, *Latine redditum.*

NUlla extat Curva cujus Æquatio ex tribus constat terminis, in qua, licet quantitates incognitæ se mutuo afficiant, vel Indices dignitatum sint surdæ quantitates (v. g. $ax^{\lambda} + bx^{\mu}y^{\sigma} + cy^{\tau} = 0$, ubi x designat Basin, y Ordinatam, λ, μ, σ, τ Indices dignitatum ipsius x & y, & a, b, c quantitates cognitas una cum signis suis + vel —) nulla inquam hujusmodi est Curva, de qua, an Quadrari possit, necne, vel quænam sint Figuræ simplicissimæ quibuscum comparari possit, sive sint Conicæ sectiones sive aliæ magis complicatæ, intra horæ octantem respondere non possim. Deinde * methodo directa & brevi, imo methodorum omnium generalium brevissima eas comparare queo. Quinetiam si duæ quævis Figuræ per hujusmodi Æquationes expressæ proponantur, per eandem Regulam, eas, modo comparari possint, comparo.

Affirmatio quidem videri potest temeraria, propterea quod perdifficile sit dictu an Figura Quadrari vel cum alia comparari possit, necne; mihi autem manifestum est, ex eo unde deduxi fonte, quanquam id aliis demonstrare in me suscipere nollem. Eadem methodus Æquationes quatuor terminorum aliasque complectitur, haud tamen adeo generaliter.

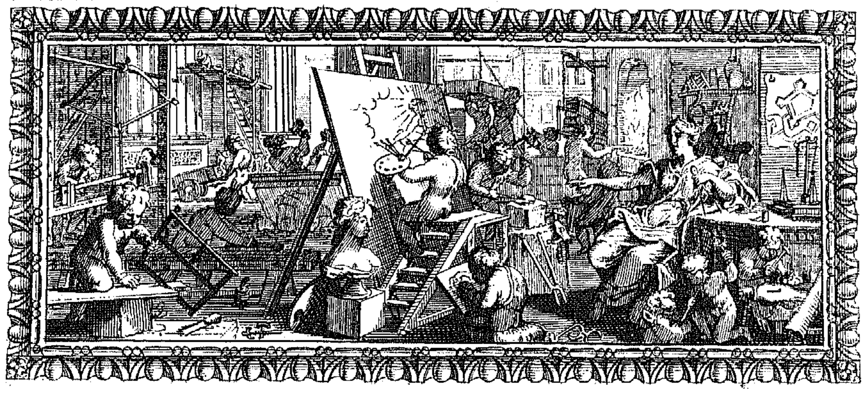
* *Methodum habes in Coroll. 2. Prop. 10. Tract. sequentis.*

TRACTATUS

DE

Quadratura Curvarum.

Pg 40 - blank page



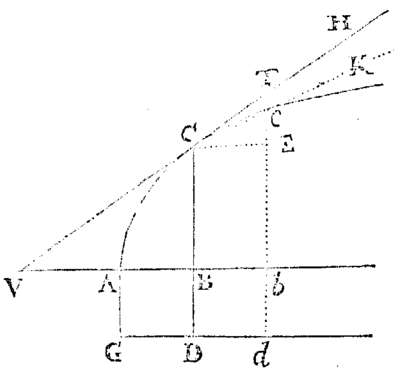
INTRODUCTIO
AD
Quadraturam Curvarum.

Quantitates Mathematicas non ut ex partibus quam minimis constantes, sed ut motu continuo descriptas hic considero. Lineæ describuntur ac describendo generantur non per appositionem partium sed per motum continuum punctorum, superficies per motum linearum, solida per motum superficierum, anguli per rotationem laterum, tempora per fluxum continuum, & sic in cæteris. Hæ Geneses in rerum natura locum vere habent & in motu corporum quotidie cernuntur. Et ad hunc modum Veteres ducendo rectas mobiles in longitudinem rectarum immobilium genesin docuerunt rectangulorum.

Considerando igitur quod quantitates æqualibus temporibus crescentes & crescendo genitæ, pro velocitate majori vel minori qua crescunt ac generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum quibus generantur; & has motuum vel incrementorum velocitates nominando *Fluxiones* & quantitates genitas nominando *Fluentes*, incidi paulatim *Annis* 1665 & 1666 in Methodum Fluxionum qua hic usus sum in Quadratura Curvarum.

Fluxiones ſunt quam proxime ut Fluentium augmenta æqualibus temporis particulis quam minimis genita, &, ut accurate loquar, ſunt in prima ratione augmentorum naſcentium; exponi autem poſſunt per lineas quaſcunque quæ ſunt ipſis proportionales.

Ut ſi areæ ABC, ABDG Ordinatis BC, BD ſuper baſi AB uniformi cum motu progredientibus deſcribantur, harum arearum fluxiones erunt inter ſe ut Ordinatæ deſcribentes BC & BD, & per Ordinatas illas exponi poſſunt, propterea quod Ordinatæ illæ ſunt ut arearum augmenta naſcentia.



Progrediatur ordinata BC de loco ſuo BC in locum quemvis novum bc. Compleatur parallelogrammum BCEb, ac ducatur recta VTH quæ Curvam tangat in C ipſiſque bc & BA productis occurrat in T & V: & Abſciſſæ AB, Ordinatæ BC, & Lineæ Curvæ ACc augmenta modo genita erunt Bb, Ec & Cc; & in horum augmentorum naſcentium ratione prima ſunt latera trianguli CET, ideoque fluxiones ipſarum AB, BC & AC ſunt ut trianguli illius CET latera CE, ET & CT & per eadem latera exponi poſſunt, vel quod perinde eſt per latera trianguli conſimilis VBC.

Eodem recidit ſi ſumantur fluxiones in ultima ratione partium evaneſcentium. Agatur recta Cc & producatur eadem ad K. Redeat Ordinata bc in locum ſuum priorem BC, & coeuntibus punctis C & c, recta CK coincidet cum tangente CH, & triangulum evaneſcens CEc in ultima ſua forma evadet ſimile triangulo CET, & ejus latera evaneſcentia CE, Ec & Cc erunt ultimo inter ſe ut ſunt trianguli alterius CET latera CE, ET & CT, & propterea in hac ratione ſunt fluxiones linearum AB, BC & AC. Si puncta C & c parvo quovis intervallo ab invicem diſtant recta CK parvo intervallo a tangente CH diſtabit. Ut recta CK cum tangente CH coincidat & rationes ultimæ linearum CE, Ec & Cc inveniantur, debent puncta C & c coire & omnino coincidere. Errores quam minimi in rebus Mathematicis non ſunt contemnendi.

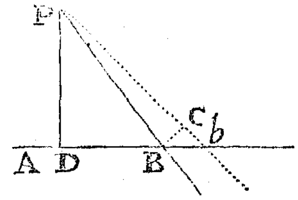
Simili argumento ſi circulus centro B radio BC deſcriptus in longitudinem Abſciſſæ AB ad angulos rectos uniformi cum motu ducatur, fluxio ſolidi geniti ABC erit ut circulus ille generans, & fluxio ſuperficiei ejus erit ut perimeter Circuli illius & fluxio lineæ curvæ AC conjunctim.

Nam

Nam quo tempore ſolidum ABC generatur ducendo circulum illum in longitudinem abſciſſæ AB, eodem ſuperficies ejus generatur ducendo perimetrum circuli illius in longitudinem Curvæ AC. Hujus methodi accipe etiam exempla quæ ſequuntur.

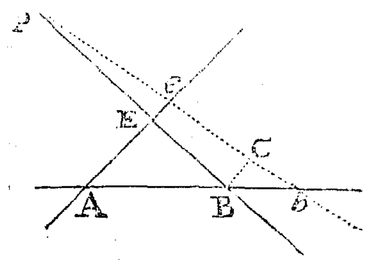
Recta PB circa polum datum P revolvens ſecet aliam poſitione datam rectam AB: quæritur proportio fluxionum rectarum illarum AB & PB.

Progrediatur recta PB de loco ſuo PB in locum novum Pb. In Pb capiatur PC ipſi PB æqualis, & ad AB ducatur PD ſic, ut angulus bPD æqualis ſit angulo bBC; & ob ſimilitudinem triangulorum bBC, bPD erit augmentum Bb ad augmentum Cb ut Pb ad Db. Redeat jam Pb in locum ſuum priorem PB ut augmenta illa evaneſcant, & evaneſcentium ratio ultima, id eſt ratio ultima Pb ad Db, ea erit quæ eſt PB ad DB, exiſtente angulo PDB recto, & propterea in hac ratione eſt fluxio ipſius AB ad fluxionem ipſius PB.



Recta PB circa datum Polum P revolvens ſecet alias duas poſitione datas rectas AB & AE in B & E: quæritur proportio fluxionum rectarum illarum AB & AE.

Progrediatur recta revolvens PB de loco ſuo PB in locum novum Pb rectas AB, AE in punctis b & e ſecantem, & rectæ AE parallela BC ducatur ipſi Pb occurrens in C, & erit Bb ad BC ut Ab ad Ae, & BC ad Ee ut PB ad PE, & conjunctis rationibus Bb ad Ee ut Ab × PB ad Ae×PE. Redeat jam linea Pb in locum ſuum priorem PB, & augmentum evaneſcens Bb erit ad augmentum evaneſcens Ee ut AB × PB ad AE × PE, ideoque in hac ratione eſt fluxio rectæ AB ad fluxionem rectæ AE.



Hinc ſi recta revolvens PB lineas quaſvis Curvas poſitione datas ſecet in punctis B & E, & rectæ jam mobiles AB, AE Curvas illas tangant in Sectionum punctis B & E: erit fluxio Curvæ quam recta AB tangit ad fluxionem Curvæ quam recta AE tangit ut AB×PB ad AE×PE. Id quod etiam eveniet ſi recta PB Curvam aliquam poſitione datam perpetuo tangat in puncto mobili P.

Fluat quantitas x *uniformiter & invenienda ſit fluxio quantitatis* x^n. Quo tempore quantitas x fluendo evadit $x + o$, quantitas x^n evadet $\overline{x+o}|^n$, id eſt per methodum ſerierum infinitarum, $x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +$ &c. Et

Et augmenta o & $nox^{n-1}+\frac{nn-n}{2}oox^{n-2}+$ &c. ſunt ad invicem ut 1 & $nx^{n-1}+\frac{nn-n}{2}ox^{n-2}+$ &c. Evaneſcant jam augmenta illa, & eorum ratio ultima erit 1 ad nx^{n-1}: ideoque fluxio quantitatis x eſt ad fluxionem quantitatis x^n ut 1 ad nx^{n-1}.

Similibus argumentis per methodum rationum primarum & ultimarum colligi poſſunt fluxiones linearum ſeu rectarum ſeu curvarum in caſibus quibuſcunque, ut & fluxiones ſuperficierum, angulorum & aliarum quantitatum. In finitis autem quantitatibus Analyſin ſic inſtituere, & finitarum naſcentium vel evaneſcentium rationes primas vel ultimas inveſtigare, conſonum eſt Geometriæ Veterum: & volui oſtendere quod in Methodo Fluxionum non opus ſit figuras infinite parvas in Geometriam introducere. Peragi tamen poteſt Analyſis in figuris quibuſcunque ſeu finitis ſeu infinite parvis quæ figuris evaneſcentibus finguntur ſimiles, ut & in figuris quæ per Methodos Indiviſibilium pro infinite parvis haberi ſolent, modo caute procedas.

Ex Fluxionibus invenire Fluentes Problema difficilius eſt, & ſolutionis primus gradus æquipollet Quadraturæ Curvarum; de qua ſequentia olim ſcripſi.

D E

Quadratura Curvarum.

Quantitates indeterminatas ut motu perpetuo creſcentes vel decreſcentes, id eſt ut fluentes vel defluentes in ſequentibus conſidero, deſignoque literis z, y, x, v, & earum fluxiones ſeu celeritates creſcendi noto iiſdem literis punctatis $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$, Sunt & harum fluxionum fluxiones ſeu mutationes magis aut minus celeres quas ipſarum z, y, x, v, fluxiones ſecundas nominare licet & ſic deſignare $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$, & harum fluxiones primas ſeu ipſarum z, y, x, v fluxiones tertias ſic $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$, & quartas ſic $\ddddot{z}$, $\ddddot{y}$, $\ddddot{x}$, $\ddddot{v}$. Et quemadmodum $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$ ſunt fluxiones quantitatum $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$, & hæ ſunt fluxiones quantitatum $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$, & hæ ſunt fluxiones quantitatum primarum z, y, x, v : ſic hæ quantitates conſiderari poſſunt ut fluxiones aliarum quas ſic deſignabo, $\overset{\prime}{z}$, $\overset{\prime}{y}$, $\overset{\prime}{x}$, $\overset{\prime}{v}$, & hæ ut fluxiones aliarum $\overset{\prime\prime}{z}$, $\overset{\prime\prime}{y}$, $\overset{\prime\prime}{x}$, $\overset{\prime\prime}{v}$, & hæ ut fluxiones aliarum $\overset{\prime\prime\prime}{z}$, $\overset{\prime\prime\prime}{y}$, $\overset{\prime\prime\prime}{x}$, $\overset{\prime\prime\prime}{v}$. Deſignant igitur $\overset{\prime\prime}{z}$, $\overset{\prime}{z}$, z, $\dot{z}$, $\ddot{z}$, $\dddot{z}$, $\ddddot{z}$, $\overset{\cdot\cdot\cdot\cdot\cdot}{z}$, &c. ſeriem quantitatum quarum quælibet poſterior eſt fluxio præcedentis & quælibet prior eſt fluens quantitas fluxionem habens ſubſequentem. Similis eſt ſeries $\overset{\prime\prime}{\sqrt{az-zz}}$, $\overset{\prime}{\sqrt{az-zz}}$, $\sqrt{az-zz}$, $\overset{\cdot}{\sqrt{az-zz}}$, $\overset{\cdot\cdot}{\sqrt{az-zz}}$, $\overset{\cdot\cdot\cdot}{\sqrt{az-zz}}$, ut & ſeries $\overset{\prime\prime}{\frac{az+zz}{a-z}}$, $\overset{\prime}{\frac{az+zz}{a-z}}$, $\frac{az+zz}{a-z}$, $\overset{\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot\cdot}{\frac{az+zz}{a-z}}$.

Et notandum eſt quod quantitas quælibet prior in his ſeriebus eſt ut area figuræ curviliniæ cujus ordinatim applicata rectangula eſt quantitas poſterior & abſciſſa eſt z: uti $\overset{\prime}{\sqrt{az-zz}}$ area curvæ cujus ordinata eſt $\sqrt{az-zz}$ & abſciſſa z. Quo autem ſpectant hæc omnia patebit in Propoſitionibus quæ ſequuntur.

PROP. I. PROB. I.

Data æquatione quotcunque fluentes quantitates involvente, invenire fluxiones.

Solutio.

Multiplicetur omnis æquationis terminus per indicem dignitatis quantitatis cujusque fluentis quam involvit, & in singulis multiplicationibus mutetur dignitatis latus in fluxionem suam, & aggregatum factorum omnium sub propriis signis erit æquatio nova.

Explicatio.

Sunto a, b, c, d &c. quantitates determinatæ & immutabiles, & proponatur æquatio quævis quantitates fluentes z, y, x &c. involvens, uti $x^3 - xy^2 + a^2z - b^3 = 0$. Multiplicentur termini primo per indices dignitatum x, & in singulis multiplicationibus pro dignitatis latere, seu x unius dimensionis, scribatur $\dot{x}$, & summa factorum erit $3\dot{x}x^2 - \dot{x}y^2$. Idem fiat in y & prodibit $-2x\dot{y}y$. Idem fiat in z & prodibit $aa\dot{z}$. Ponatur summa factorum æqualis nihilo, & habebitur æquatio $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Dico quod hac æquatione definitur relatio fluxionum.

Demonstratio.

Nam sit o quantitas admodum parva & sunto $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, quantitatum z, y, x momenta id est incrementa momentanea synchrona. Et si quantitates fluentes jam sunt z, y & x hæ post momentum temporis incrementis suis $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ auctæ, evadent $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$, quæ in æquatione prima pro z, y, & x scriptæ dant æquationem $x^3 + 3x^2o\dot{x} + 3xo^2\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xo\dot{y}y - 2\dot{x}o^2\dot{y}y - xo^2\dot{y}\dot{y} - \dot{x}o^3\dot{y}\dot{y} + a^2z + a^2o\dot{z} - b^3 = 0$.

Subducatur æquatio prior, & residuum divisum per o erit $3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3o^2 - \dot{x}y^2 - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} - \dot{x}o^2\dot{y}\dot{y} + a^2\dot{z} = 0$. Minuatur quantitas o in infinitum, & neglectis terminis evanescentibus restabit $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Q. E. D.

Explicatio

Explicatio plenior.

Ad eundem modum si æquatio esset $x^3 - xy^2 + a^2\sqrt{ax-yy} - b^3 = 0$, produceretur $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{\overline{\sqrt{ax-yy}}} = 0$. Ubi si fluxionem $\dot{\overline{\sqrt{ax-yy}}}$ tollere velis, pone $\sqrt{ax-yy} = z$, & erit $ax - y^2 = z^2$, & (per hanc Propositionem) $a\dot{x} - 2\dot{y}y = 2\dot{z}z$ seu $\frac{a\dot{x}-2\dot{y}y}{2z} = \dot{z}$, hoc est $\frac{a\dot{x}-2\dot{y}y}{2\sqrt{ax-yy}} = \dot{\overline{\sqrt{ax-yy}}}$. Et inde $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + \frac{a^3\dot{x}-2a^2\dot{y}y}{2\sqrt{ax-yy}} = 0$.

Et per operationem repetitam pergitur ad fluxiones secundas, tertias & sequentes. Sit æquatio $zy^3 - z^4 + a^4 = 0$, & fiet per operationem primam $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$, per secundam $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 4\ddot{z}z^3 - 12\dot{z}^2z^2 = 0$, per tertiam $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$.

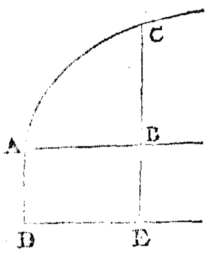
Ubi vero sic pergitur ad fluxiones secundas, tertias & sequentes, convenit quantitatem aliquam ut uniformiter fluentem considerare, & pro ejus fluxione prima unitatem scribere, pro secunda vero & sequentibus nihil. Sit æquatio $zy^3 - z^4 + a^4 = 0$, ut supra; & fluat z uniformiter, sitque ejus fluxio unitas, & fiet per operationem primam $y^3 + 3z\dot{y}y^2 - 4z^3 = 0$, per secundam $6\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 12z^2 = 0$, per tertiam $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z = 0$.

In hujus autem generis æquationibus concipiendum est quod fluxiones in singulis terminis sint ejusdem ordinis, id est vel omnes primi ordinis $\dot{y}$, $\dot{z}$, vel omnes secundi $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$, vel omnes tertii $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, &c. Et ubi res aliter se habet complendus est ordo per subintellectas fluxiones quantitatis uniformiter fluentis. Sic æquatio novissima complendo ordinem tertium fit $9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24\dot{z}z^3 = 0$.

PROP.

PROP. II. PROB. II.

Invenire Curvas quæ quadrari possunt.



Sit ABC figura invenienda, BC Ordinatim applicata rectangula, & AB abscissa. Producatur CB ad E ut sit BE = 1, & compleatur parallelogrammum ABED: & arearum ABC, ABED fluxiones erunt ut BC & BE. Assumatur igitur æquatio quævis qua relatio arearum definiatur, & inde dabitur relatio ordinatarum BC & BE per Prop. 1. Q. E. I.

Hujus rei exempla habentur in Propositionibus duabus sequentibus.

PROP. III. THEOR. I.

Si pro abscissa AB & area AE seu AB × 1 promiscue scribatur z, & si pro $e + fz^{n} + gz^{2n} + hz^{3n}$ + &c. scribatur R: sit autem area Curvæ $z^{\theta}R^{\lambda}$, erit ordinatim applicata BC æqualis

$$\overline{\theta e + \begin{matrix}\theta\\ +\lambda n\end{matrix} \times fz^{n} + \begin{matrix}\theta\\ +2\lambda n\end{matrix} \times gz^{2n} + \begin{matrix}\theta\\ +3\lambda n\end{matrix} \times hz^{3n} + \&c.} \times z^{\theta-1}R^{\lambda-1}.$$

Demonstratio.

Nam si sit $z^{\theta}R^{\lambda} = v$, erit (per Prop. 1) $\theta \dot{z} z^{\theta-1} R^{\lambda} + \lambda z^{\theta} \dot{R} R^{\lambda-1} = \dot{v}$. Pro R^{λ} in primo æquationis termino & z^{θ} in secundo scribe $RR^{\lambda-1}$ & $zz^{\theta-1}$, & fiet $\theta \dot{z} R + \lambda z \dot{R}$ in $z^{\theta-1}R^{\lambda-1} = \dot{v}$. Erat autem $R = e + fz^{n} + gz^{2n} + hz^{3n}$ + &c. et inde (per Prop. 1) fit $\dot{R} = nf\dot{z}z^{n-1} + 2ng\dot{z}z^{2n-1} + 3nh\dot{z}z^{3n-1}$ + &c. quibus substitutis & scripta BE seu 1 pro $\dot{z}$, fiet

$$\overline{\theta e + \begin{matrix}\theta\\ +\lambda n\end{matrix} \times fz^{n} + \begin{matrix}\theta\\ +2\lambda n\end{matrix} \times gz^{2n} + \begin{matrix}\theta\\ +3\lambda n\end{matrix} \times hz^{3n} + \&c.} \times z^{\theta-1}R^{\lambda-1} = \dot{v} = BC.$$

Q. E. D.

PROP.

PROP. IV. THEOR. II.

Si Curvæ abſciſſa AB ſit z, & ſi pro $e+fz^{n}+gz^{2n}+$ &c. ſcribatur R, & pro $k+lz^{n}+mz^{2n}+$ &c. ſcribatur S; ſit autem area Curvæ $z^{\theta}R^{\lambda}S^{\mu}$: Erit ordinatim applicata BC æqualis

$$\left.\begin{array}{llll} \theta ek + \overline{\theta+\lambda n}\times fkz^{n} & +\overline{\theta+2\lambda n}\times gkz^{2n} & \ldots * \ldots\ldots & * \ldots \\ +\overline{\theta+\mu n}\times elz^{n} & +\overline{\theta+\lambda n+\mu n}\times flz^{2n} & +\overline{\theta+2\lambda n+\mu n}\times glz^{3n} & \ldots * \ldots \\ & +\overline{\theta+2\mu n}\times emz^{2n} & +\overline{\theta+\lambda n+2\mu n}\times fmz^{3n} & +\overline{\theta+2\lambda n+2\mu n}\times gmz^{4n} \end{array}\right\}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$$

Demonſtratur ad modum Propoſitionis ſuperioris.

PROP. V. THEOR. III.

Si Curvæ abſciſſa AB ſit z, & pro $e+fz^{n}+gz^{2n}+hz^{3n}+$ &c. ſcribatur R: ſit autem Ordinatim applicata $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^{n}+cz^{2n}+dz^{3n}+\&c.}$ & ponatur $\frac{\theta}{n}=r$, $r+\lambda=s$, $s+\lambda=t$, $t+\lambda=v$, &c.

$$\text{Erit Area}=z^{\theta}R^{\lambda}\text{ in }\begin{array}{l} +\dfrac{\frac{1}{n}a}{re} \\ +\dfrac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}z^{n} \\ +\dfrac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}z^{2n} \\ +\dfrac{\frac{1}{n}d-\overline{s+2}\times fC-\overline{t+1}\times gB-vhA}{\overline{r+3}\times e}z^{3n} \\ +\dfrac{-\overline{s+3}\times fD-\overline{t+2}\times gC-\overline{v+1}\times hB}{\overline{r+4}\times e}z^{4n} \\ +\ \&c. \end{array}$$

Ubi A, B, C, D, &c. denotant totas coefficientes datas terminorum ſingulorum in ſerie cum ſignis ſuis $+$ & $-$, nempe

A primi termini coefficientem $\dfrac{\frac{1}{n}a}{re}$

B ſecundi termini coefficientem $\dfrac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}$

C tertii termini coefficientem $\dfrac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}$

Et ſic deinceps.

Demonstratio.

Sunto juxta Propositionem tertiam.

$$
\begin{array}{lllll}
 & \textsc{Curvarum Ordinatæ} & & & \textsc{Areæ} \\
1 & \theta eA + \overline{\theta+\lambda\eta}\times fAz^{\eta} + \overline{\theta+2\lambda\eta}\times gAz^{2\eta} + \overline{\theta+3\lambda\eta}\times hAz^{3\eta},\ \&c. & \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} & \times z^{\theta-1}R^{\lambda-1}. & \left\{\begin{array}{l} Az^{\theta}R^{\lambda} \\ Bz^{\theta+\eta}R^{\lambda} \\ Cz^{\theta+2\eta}R^{\lambda} \\ Dz^{\theta+3\eta}R^{\lambda} \end{array}\right. \\
2 & \ldots + \overline{\theta+\eta}\times eBz^{\eta} + \overline{\theta+\eta+\lambda\eta}\times fBz^{2\eta} + \overline{\theta+\eta+2\lambda\eta}\times gBz^{3\eta}\ \&c. & & & \\
3 & \ldots\ldots + \overline{\theta+2\eta}\times eCz^{2\eta} + \overline{\theta+2\eta+\lambda\eta}\times fCz^{3\eta}\ \&c. & & & \\
4 & \ldots\ldots\ldots + \overline{\theta+3\eta}\times eDz^{3\eta}\ \&c. & & &
\end{array}
$$

Et si summa Ordinatarum ponatur æqualis Ordinatæ $a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} +$ &c. in $z^{\theta-1}R^{\lambda-1}$, summa arearum $z^{\theta}R^{\lambda}$ in $A + Bz^{\eta} + Cz^{2\eta} + Dz^{3\eta} +$ &c. æqualis erit areæ Curvæ cujus ista est Ordinata. Æquentur igitur Ordinatarum termini correspondentes,

et fiet $a = \theta eA$,

$b = \overline{\theta+\lambda\eta}\times fA + \overline{\theta+\eta}\times eB$,

$c = \overline{\theta+2\lambda\eta}\times gA + \overline{\theta+\eta+\lambda\eta}\times fB + \overline{\theta+2\eta}\times eC$,

&c.

et inde $A = \frac{a}{\theta e}$

$B = \frac{b - \overline{\theta+\lambda\eta}\times fA}{\overline{\theta+\eta}\times e}$

$C = \frac{c - \overline{\theta+2\lambda\eta}\times gA - \overline{\theta+\eta+\lambda\eta}\times fB}{\overline{\theta+2\eta}\times e}$

Et sic deinceps in infinitum.

Pone jam $\frac{\theta}{\eta} = r$, $r + \lambda = s$, $s + \lambda = t$, &c. et in area $z^{\theta}R^{\lambda}$ in $A + Bz^{\eta} + Cz^{2\eta} + Dz^{3\eta} +$ &c. scribe ipsorum A, B, C, &c. valores inventos, & prodibit series proposita. Q. E. D.

Et notandum est quod Ordinata omnis duobus modis in seriem resolvitur. Nam index η vel affirmativus esse potest vel negativus.

Proponatur Ordinata $\frac{3k - lzz}{zz\sqrt{kz - lz^3 + mz^4}}$: Hæc vel sic scribi potest

$z^{-\frac{5}{2}} \times \overline{3k - lz^2} \times \overline{k - lz^2 + mz^3}|^{-\frac{1}{2}}$,

vel sic $z^{-2} \times \overline{-l + 3kz^{-2}} \times \overline{m - lz^{-1} + kz^{-3}}|^{-\frac{1}{2}}$.

In

In caſu priore eſt $a=3k$, $b=0$, $c=-l$, $e=k$, $f=0$, $g=-l$, $h=m$, $\lambda=\frac{1}{2}$, $n=1$, $\theta-1=-\frac{3}{2}$, $\theta=-\frac{1}{2}=r$, $s=-1$, $t=-\frac{1}{2}$, $v=0$.

In poſteriore eſt $a=-l$, $b=0$, $c=3k$, $e=m$, $f=-l$, $g=0$, $h=k$, $\lambda=\frac{1}{2}$, $n=-1$, $\theta-1=-2$, $\theta=-1$, $r=1$, $s=1\frac{1}{2}$, $t=2$, $v=2\frac{1}{2}$. Tentandus eſt caſus uterque. Et ſi ſerierum alterutra ob terminos tandem deficientes abrumpitur ac terminatur habebitur area Curvæ in terminis finitis. Sic in exempli hujus priore caſu ſcribendo in ſerie valores ipſorum a, b, c, e, f, g, h, λ, θ, r, s, t, v, termini omnes poſt primum evaneſcunt in infinitum & area Curvæ prodit $-2\sqrt{\frac{k-lz^2+mz^3}{z^3}}$. Et hæc area ob ſignum negativum adjacet abſciſſæ ultra ordinatam productæ. Nam area omnis affirmativa adjacet tam abſciſſæ quam ordinatæ, negativa vero cadit ad contrarias partes ordinatæ & adjacet abſciſſæ productæ, manente ſcilicet ſigno Ordinatæ. Hoc modo ſeries alterutra & nonnunquam utraque ſemper terminatur & finita evadit ſi Curva geometrice quadrari poteſt. At ſi Curva talem quadraturam non admittit, ſeries utraque continuabitur in infinitum, & earum altera converget & aream dabit approximando, præterquam ubi r (propter aream infinitam) vel nihil eſt vel numerus integer & negativus, vel ubi $\frac{z^n}{e}$ æqualis eſt unitati. Si $\frac{z^n}{e}$ minor eſt unitate, converget ſeries in qua index n affirmativus eſt: ſin $\frac{z^n}{e}$ unitate major eſt, converget ſeries altera. Si in uno caſu area adjacet abſciſſæ ad uſque ordinatam ductæ, in altero adjacet abſciſſæ ultra ordinatam productæ.

Nota inſuper quod ſi Ordinata contentum eſt ſub factore rationali Q & factore ſurdo irreducibili R^{π} & factoris ſurdi latus R non dividit factorem rationalem Q; erit $\lambda-1=\pi$ & $R^{\lambda-1}=R^{\pi}$. Sin factoris ſurdi latus R dividit factorem rationalem ſemel, erit $\lambda-1=\pi+1$ & $R^{\lambda-1}=R^{\pi+1}$: ſi dividit bis, erit $\lambda-1=\pi+2$ & $R^{\lambda-1}=R^{\pi+2}$: ſi ter, erit $\lambda-1=\pi+3$, & $R^{\lambda-1}=R^{\pi+3}$: & ſic deinceps.

Si Ordinata eſt fractio rationalis irreducibilis cum denominatore ex duobus vel pluribus terminis compoſito: reſolvendus eſt denominator in diviſores ſuos omnes primos. Et ſi diviſor ſit aliquis cui nullus alius eſt æqualis, Curva quadrari nequit: Sin duo vel plures ſint diviſores æquales, rejiciendus eſt eorum unus, & ſi adhuc alii duo vel plures ſint ſibi mutuo æquales & prioribus inæquales, rejiciendus eſt etiam eorum unus, & ſic in aliis omnibus æqualibus ſi adhuc plures ſint: deinde diviſor qui relinquitur vel contentum ſub diviſoribus omnibus qui relinquuntur, ſi plures ſunt, ponendum eſt pro R, & ejus quadrati reciprocum R^{-2} pro $R^{\lambda-1}$, præterquam ubi contentum illud eſt quadratum vel cubus vel quadrato quadratum, &c. quo caſu ejus latus ponendum eſt pro R & poteſtatis index 2 vel 3 vel 4 negative ſumptus pro λ, et Ordinata ad denominatorem R^2 vel R^3 vel R^4 vel R^5 &c. reducenda.

Ut

Ut ſi Ordinata ſit $\frac{z^5+z^4-8z^3}{z^5+z^4-5z^3-z^2+8z-4}$; quoniam hæc fractio irreducibilis eſt & denominatoris diviſores ſunt pares, nempe $z-1$, $z-1$, $z-1$ & $z+2$, $z+2$, rejicio magnitudinis utriuſque diviſorem unum, & reliquorum $z-1$, $z-1$, $z+2$ contentum z^3-3z+2 pono pro R & ejus quadrati reciprocum $\frac{1}{R^2}$ ſeu R^{-2} pro $R^{\lambda-1}$. Dein Ordinatam ad denominatorem R^2 ſeu $R^{1-\lambda}$ reduco, & fit $\frac{z^6-9z^4+8z^3}{\overline{z^3-3z+2}|^2}$, i.e. $z^3 \times \overline{8-9z+z^3} \times \overline{2-3z+z^3}|^{-2}$. Et inde eſt $a=8$, $b=-9$, $c=0$, $d=1$, &c. $e=2$, $f=-3$, $g=0$, $h=1$, $\lambda-1=-2$, $\lambda=-1$, $\eta=1$, $\theta-1=3$, $\theta=4=r$, $s=3$, $t=2$, $v=1$. Et his in ſerie ſcriptis prodit area $\frac{z^4}{z^3-3z+2}$, terminis omnibus in tota ſerie poſt primum evaneſcentibus.

Si denique Ordinata eſt fractio irreducibilis & ejus denominator contentum eſt ſub factore rationali Q & factore ſurdo irreducibili R^π, inveniendi ſunt lateris R diviſores omnes primi, & rejiciendus eſt diviſor unus magnitudinis cujuſque & per diviſores qui reſtant, ſiqui ſint, multiplicandus eſt factor rationalis Q: & ſi factum æquale eſt lateri R vel lateris illius poteſtati alicui cujus index eſt numerus integer, eſto index ille m, & erit $\lambda-1=-\pi-m$, & $R^{\lambda-1}=R^{-\pi-m}$.

Ut ſi Ordinata ſit $\frac{3q^5-q^4x+9q^2x^2-q^2x^3-6x^3}{\overline{q^2-x^2} \times \overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$, quoniam factoris ſurdi latus R ſeu $q^3+q^2x-qx^2-x^3$ diviſores habet $q+x$, $q+x$, $q-x$ qui duarum ſunt magnitudinum, rejicio diviſorem unum magnitudinis utriuſque & per diviſorem $q+x$ qui relinquitur, multiplico factorem rationalem q^2-x^2. Et quoniam factum $q^3+q^2x-qx^2-x^3$ æquale eſt lateri R, pono $m=1$, & inde, cum π ſit $\frac{1}{3}$, ſit $\lambda-1=-\frac{4}{3}$. Ordinatam igitur reduco ad denominatorem $R^{-\frac{4}{3}}$, & fit

$$x^0 \times \overline{3q^6+2q^5x+8q^4x^2+8q^3x^3-7q^2x^4-6qx^5} \times \overline{q^3+q^2x-qx^2-x^3}|^{-\frac{4}{3}}$$

Unde eſt $a=3q^6$, $b=2q^5$ &c. $e=q^3$, $f=q^2$ &c. $\theta-1=0$, $\theta=1=\eta$, $\lambda=-\frac{1}{3}$, $r=1$, $s=\frac{2}{3}$, $t=\frac{1}{3}$, $v=0$. Et his in ſerie ſcriptis prodit area $\frac{3q^2x+3x^3}{\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$, terminis omnibus in ſerie tota poſt tertium evaneſcentibus.

PROP. VI. THEOR. IV.

Si Curvæ abſciſſa AB ſit z, & ſcribantur R pro $e+fz^\eta+gz^{2\eta}+hz^{3\eta}+$&c. & S pro $k+lz^\eta+mz^{2\eta}+nz^{3\eta}+$&c. ſit autem Ordinatim applicata $z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ in $a+bz^\eta+cz^{2\eta}+dz^{3\eta}+$&c. et ſi terminorum, e, f, g, h, &c. et k, l, m, n, &c. rectangula ſint

ek	fk	gk	hk	&c.
el	fl	gl	hl	&c.
em	fm	gm	hm	&c.
en	fn	gn	hn	&c.

Et

Et fi rectangulorum illorum coefficientes numerales fint refpective

$$
\begin{array}{lllll}
\frac{\theta}{n} = r. & r + \lambda = s. & s + \lambda = t. & t + \lambda = v. & \&c. \\
r + \mu = s'. & s + \mu = t'. & t + \mu = v'. & v + \mu = w'. & \&c. \\
s' + \mu = t''. & t' + \mu = v''. & v' + \mu = w''. & w' + \mu = x''. & \&c. \\
t'' + \mu = v'''. & v'' + \mu = w'''. & w'' + \mu = x'''. & x'' + \mu = y'''. & \&c.
\end{array}
$$

Area Curvæ erit hæc

$$
\begin{aligned}
z^{\theta} R^{\lambda} S^{\mu} \text{ in } & + \frac{\frac{1}{n} a}{rek} \\
& + \frac{\frac{1}{n} b \begin{smallmatrix} -sfk \\ -s'el \end{smallmatrix} A}{\overline{r+1} \times ek} z^{n} \\
& + \frac{\frac{1}{n} c \begin{smallmatrix} -\overline{s+1} \times fk \\ -\overline{s'+1} \times el \end{smallmatrix} B \begin{smallmatrix} -tgk \\ -t'fl \\ -t''em \end{smallmatrix} A}{\overline{r+2} \times ek} z^{2n} \\
& + \frac{\frac{1}{n} d \begin{smallmatrix} -\overline{s+2} \times fk \\ -\overline{s'+2} \times el \end{smallmatrix} C \begin{smallmatrix} -\overline{t+1} \times gk \\ -\overline{t'+1} \times fl \\ -\overline{t''+1} \times em \end{smallmatrix} B \begin{smallmatrix} -vbk \\ -v'gl \\ -v''fm \\ -v'''en \end{smallmatrix} A}{\overline{r+3} \times ek} z^{3n} \\
& + \&c.
\end{aligned}
$$

Ubi A denotat termini primi coefficientem datam $\frac{\frac{1}{n} a}{rek}$ cum figno fuo $+$ vel $-$, B coefficientem datam fecundi, C coefficientem datam tertii, & fic deinceps. Terminorum vero, a, b, c, &c. e, f, g, &c. k, l, m, &c. unus vel plures deeffe poffunt.

Demonftratur Propofitio ad modum præcedentis, & quæ ibi notantur hic obtinent. Pergit autem feries talium Propofitionum in infinitum, & progreffio feriei manifefta eft.

PROP. VII. THEOR. V.

Si pro $e + fz^{n} + gz^{2n} +$ &c. fcribatur R ut fupra, & in Curvæ alicujus Ordinata $z^{\theta \pm n\sigma} R^{\lambda \pm \tau}$ maneant quantitates datæ θ, n, λ, e, f, g, &c. et pro σ ac τ fcribantur fucceffive numeri quicunque integri: & fi detur Area unius ex Curvis quæ per Ordinatas innumeras fic prodeuntes defignantur fi Ordinatæ funt duorum nominum in vinculo radicis, vel fi dentur Areæ duarum ex Curvis fi Ordinatæ funt trium nominum in vinculo radicis, vel Areæ trium ex Curvis fi Ordinatæ funt quatuor nominum in vinculo radicis, & fic deinceps in infinitum: dico quod dabuntur Areæ curvarum omnium.

Pro nominibus hic habeo terminos omnes in vinculo radicis tam deficientes quam plenos quorum indices dignitatum sunt in progressione arithmetica. Sic Ordinata $\sqrt{a^4 - ax^3 + x^4}$ ob terminos duos inter a^4 & $-ax^3$ deficientes pro quinquinomio haberi debet. At $\sqrt{a^4 + x^4}$ binomium est, & $\sqrt{a^4 + x^4 - \frac{x^8}{a^4}}$ trinomium, cum progressio jam per majores differentias procedat. Propositio vero sic demonstratur.

CAS. I.

Sunto Curvarum duarum Ordinatæ $pz^{\theta-1}R^{\lambda-1}$ & $qz^{\theta+n-1}R^{\lambda-1}$, & Areæ pA & qB, existente R quantitate trium nominum $e + fz^n + gz^{2n}$. Et cum per Prop. 3. sit $z^\theta R^\lambda$ area Curvæ cujus Ordinata est $\theta e + \overline{\theta + \lambda n} \times fz^n + \overline{\theta + 2\lambda n} \times gz^{2n}$ in $z^{\theta-1}R^{\lambda-1}$, subduc Ordinatas & Areas priores de Area & Ordinata posteriori, & manebit $\overline{\theta e - p + \overline{\begin{smallmatrix}\theta\\+\lambda n\end{smallmatrix}} \times f - q \times z^n + \begin{smallmatrix}\theta\\+2\lambda n\end{smallmatrix} \times gz^{2n}} \times z^{\theta-1}R^{\lambda-1}$ Ordinata nova Curvæ, et $z^\theta R^\lambda - pA - qB$ ejusdem Area. Pone $\theta e = p$, & $\theta f + \lambda nf = q$, & Ordinata evadet $\overline{\theta + 2\lambda n} \times gz^{2n} \times z^{\theta-1}R^{\lambda-1}$, & Area $z^\theta R^\lambda - \theta eA - \theta fB - \lambda nfB$. Divide utramque per $\theta g + 2\lambda ng$, & Aream prodeuntem dic C, & assumpta utcunque r, erit rC Area Curvæ cujus Ordinata est $rz^{\theta+2n-1}R^{\lambda-1}$. Et qua ratione ex Areis pA & qB Aream rC Ordinatæ $rz^{\theta+2n-1}R^{\lambda-1}$ congruentem invenimus, licebit ex Areis qB & rC Aream quartam puta sD, Ordinatæ $sz^{\theta+3n-1}R^{\lambda-1}$ congruentem invenire, & sic deinceps in infinitum. Et par est ratio progressionis ab Areis B & A in partem contrariam pergentis. Si terminorum θ, $\theta + \lambda n$, & $\theta + 2\lambda n$ aliquis deficit & seriem abrumpit, assumatur Area pA in principio progressionis unius & Area qB in principio alterius, & ex his duabus Areis dabuntur Areæ omnes in progressione utraque. Et contra, ex aliis duabus Areis assumptis fit regressus per Analysin ad Areas A & B, adeo ut ex duabus datis cæteræ omnes dentur. Q. E. O.

Hic est casus Curvarum ubi ipsius z index θ augetur vel diminuitur perpetua additione vel subductione quantitatis n. Casus alter est Curvarum ubi index λ augetur vel diminuitur unitatibus.

CAS. II.

Ordinatæ $pz^{\theta-1}R^\lambda$ et $qz^{\theta+n-1}R^\lambda$, quibus Areæ pA et qB jam respondeant, si in R seu $e + fz^n + gz^{2n}$ ducantur ac deinde ad R vicissim applicentur, evadunt $\overline{pe + pfz^n + pgz^{2n}} \times z^{\theta-1}R^{\lambda-1}$, et $\overline{qez^n + qfz^{2n} + qgz^{3n}} \times z^{\theta-1}R^{\lambda-1}$.

Et (per Prop. 3.) est $az^\theta R^\lambda$ area Curvæ cujus Ordinata est $\overline{\theta ae + \overline{\theta + \lambda n} \times afz^n + \overline{\theta + 2\lambda n} \times agz^{2n}} \times z^{\theta-1}R^{\lambda-1}$, et $bz^{\theta+n}R^\lambda$ area Curvæ

cujus

cujus Ordinata est $\overline{\overline{\theta+n} \times bez^{n} + \overline{\theta+n+\lambda n} \times bfz^{2n} + \overline{\theta+n+2\lambda n} \times bgz^{3n}} \times z^{\theta-1} R^{\lambda-1}$.

Et harum quatuor Arearum summa est $pA + qB + az^{\theta} R^{\lambda} + bz^{\theta+n} R^{\lambda}$, & summa respondentium Ordinatarum

$$\begin{array}{llll} \theta ae + \overline{\theta+\lambda n} \times afz^{n} + & \overline{\theta+2\lambda n} \times agz^{2n} + & \overline{\theta+n+2\lambda n} \times bgz^{3n} & \times z^{\theta-1} R^{\lambda-1}. \\ + pe + \overline{\theta+n} \times be & + \overline{\theta+n+\lambda n} \times bf & + 1 \times qg & \\ \quad + 1 \times pf & \quad + 1 \times pg & & \\ \quad + 1 \times qe & \quad + 1 \times qf & & \end{array}$$

Si terminus primus tertius & quartus ponantur seorsim æquales nihilo, per primum fiet $\theta ae + pe = 0$, seu $-\theta a = p$, per quartum $-\theta b - nb - 2\lambda nb = q$, & per tertium (eliminando p & q) $\frac{2ag}{f} = b$. Unde secundus fit $\frac{\lambda naff - 4\lambda nage}{f}$, adeoq; summa quatuor Ordinatarum est $\frac{\lambda naff - 4\lambda nage}{f} z^{\theta+n-1} R^{\lambda-1}$, & summa totidem respondentium Arearum est $az^{\theta} R^{\lambda} + \frac{2ag}{f} z^{\theta+n} R^{\lambda} - \theta aA + \frac{2\theta+2n+4\lambda n}{f} agB$. Dividantur hæ summæ per $\frac{\lambda naff - 4\lambda nage}{f}$, & si Quotum posterius dicatur D, erit D Area curvæ cujus Ordinata est Quotum prius $z^{\theta+n-1} R^{\lambda-1}$. Et eadem ratione ponendo omnes Ordinatæ terminos præter primum æquales nihilo potest Area Curvæ inveniri cujus Ordinata est $z^{\theta-1} R^{\lambda-1}$. Dicatur Area ista C, & qua ratione ex Areis A & B inventæ sunt Areæ C ac D, ex his Areis C ac D inveniri possunt aliæ duæ E & F Ordinatis $z^{\theta-1} R^{\lambda-2}$ et $z^{\theta+n-1} R^{\lambda-2}$ congruentes, & sic deinceps in infinitum. Et per Analysin contrariam regredi licet ab Areis E & F ad Areas C ac D, & inde ad Areas A & B, aliasque quæ in progressione sequuntur. Igitur si index λ perpetua unitatum additione vel subductione augeatur vel minuatur, & ex Areis quæ Ordinatis sic prodeuntibus respondent duæ simplicissimæ habentur; dantur aliæ omnes in infinitum. Q. E. O.

CAS. III.

Et per casus hosce duos conjunctos, si tam index θ perpetua additione vel subductione ipsius n, quam index λ perpetua additione vel subductione unitatis, utcunque augeatur vel minuatur, dabuntur Areæ singulis prodeuntibus Ordinatis respondentes. Q. E. O.

CAS. IV.

Et simili argumento si Ordinata constat ex quatuor nominibus in vinculo radicali & dantur tres Arearum, vel si constat ex quinque nominibus & dantur quatuor Arearum, & sic deinceps: dabuntur Areæ omnes quæ addendo

dendo vel subducendo numerum n indici θ vel unitatem indici λ generari possunt. Et par est ratio Curvarum ubi Ordinatæ ex binomiis constantur, & Area una earum quæ non sunt Geometrice quadrabiles datur. Q. E. O.

PROP. VIII. THEOR. VI.

Si pro $e + fz^n + gz^{2n} + \&c.$ et $k + lz^n + mz^{2n} + \&c.$ scribantur R & S ut supra, & in Curvæ alicujus Ordinata $z^{\theta \pm n\sigma} R^{\lambda \pm \tau} S^{\mu \pm \upsilon}$ maneant quantitates datæ θ, n, λ, μ, e, f, g, k, l, m, &c. et pro σ, τ, & υ, scribantur successive numeri quicunque integri: & si dentur Areæ duarum ex curvis quæ per Ordinatas sic prodeuntes designantur si quantitates R & S sunt binomia, vel si dentur Areæ trium ex curvis si R & S conjunctim ex quinque nominibus constant, vel Areæ quatuor ex curvis si R & S conjunctim ex sex nominibus constant, & sic deinceps in infinitum: dico quod dabuntur Areæ curvarum omnium.

Demonstratur ad modum propositionis superioris.

PROP. IX. THEOR. VII.

Æquantur Curvarum Areæ inter se quarum Ordinatæ sunt reciproce ut fluxiones Abscissarum.

Nam contenta sub Ordinatis & fluxionibus Abscissarum erunt æqualia, & fluxiones Arearum sunt ut hæc contenta.

COROL. I.

Si assumatur relatio quævis inter Abscissas duarum Curvarum, & inde per Prop. I. quæratur relatio fluxionum Abscissarum, & ponantur Ordinatæ reciproce proportionales fluxionibus, inveniri possunt innumeræ Curvæ quarum Areæ sibi mutuo æquales erunt.

COROL. II.

Si enim Curva omnis cujus hæc est Ordinata $z^{\theta-1} \times \overline{e + fz^n + gz^{2n} + \&c.}|^{\lambda}$, assumendo quantitatem quamvis pro ν, & ponendo $\frac{n}{\nu} = s$, & $z^s = x$, migrat in aliam sibi æqualem cujus Ordinata est $\frac{\nu}{n} x^{\frac{\nu\theta - n}{n}} \times \overline{e + fx^\nu + gx^{2\nu} + \&c.}|^{\lambda}$.

COROL.

COROL. III.

Et Curva omnis cujus Ordinata est $z^{\vartheta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}$, assumendo quantitatem quamvis pro ν & ponendo $\frac{\eta}{\nu}=s$, & $z^{s}=x$, migrat in aliam sibi æqualem cujus Ordinata est $\frac{\nu}{\eta}x^{\frac{\nu\vartheta-\eta}{\eta}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$.

COROL. IV.

Et Curva omnis cujus Ordinata est $z^{\vartheta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}\times\overline{k+lz^{\eta}+mz^{2\eta}+\&c.}|^{\mu}$, assumendo quantitatem quamvis pro ν, & ponendo $\frac{\eta}{\nu}=s$, & $z^{s}=x$, migrat in aliam sibi æqualem cujus Ordinata est

$$\frac{\nu}{\eta}x^{\frac{\nu\vartheta-\eta}{\eta}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}\times\overline{k+lx^{\nu}+mx^{2\nu}+\&c.}|^{\mu}$$

COROL. V.

Et Curva omnis cujus Ordinata est $z^{\vartheta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}$, ponendo $\frac{1}{z}=x$, migrat in aliam sibi æqualem cujus Ordinata est $\frac{1}{x^{\vartheta+1}}\times\overline{e+fx^{-\eta}+gx^{-2\eta}+\&c.}|^{\lambda}$ id est $\frac{1}{x^{\vartheta+1+\eta\lambda}}\times\overline{f+ex^{\eta}}|^{\lambda}$ si duo sunt nomina in vinculo radicis, vel $\frac{1}{x^{\vartheta+1+2\eta\lambda}}\times\overline{g+fx^{\eta}+ex^{2\eta}}|^{\lambda}$ si tria sunt nomina; & sic deinceps.

COROL. VI.

Et Curva omnis cujus Ordinata est $z^{\vartheta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}\times\overline{k+lz^{\eta}+mz^{2\eta}+\&c.}|^{\mu}$, ponendo $\frac{1}{z}=x$, migrat in aliam sibi æqualem cujus Ordinata est

$$\frac{1}{x^{\vartheta+1}}\times\overline{e+fx^{-\eta}+gx^{-2\eta}+\&c.}|^{\lambda}\times\overline{k+lx^{-\eta}+mx^{-2\eta}+\&c.}|^{\mu}$$

id est $\frac{1}{x^{\vartheta+1+\eta\lambda+\eta\mu}}\times\overline{f+ex^{\eta}}|^{\lambda}\times\overline{l+kx^{\eta}}|^{\mu}$ si bina sunt nomina in vinculis radicum, vel $\frac{1}{x^{\vartheta+1+2\eta\lambda+\eta\mu}}\times\overline{g+fx^{\eta}+ex^{2\eta}}|^{\lambda}\times\overline{l+kx^{\eta}}|^{\mu}$ si tria sunt nomina in vinculo radicis prioris ac duo in vinculo posterioris: & sic in aliis.

Et nota quod Areæ duæ æquales in noviſſimis hiſce duobus Corollariis jacent ad contrarias partes Ordinatarum. Si Area in alterutra Curva adjacet Abſciſſæ, Area huic æqualis in altera Curva adjacet Abſciſſæ productæ.

COROL. VII.

Si relatio inter Curvæ alicujus Ordinatam y & Abſciſſam z definiatur per æquationem quamvis affectam hujus formæ,

$$y^{\alpha}\times\overline{e+fy^{n}z^{\delta}+gy^{2n}z^{2\delta}+hy^{3n}z^{3\delta}+\&c.}=z^{\beta}\times\overline{k+ly^{n}z^{\delta}+my^{2n}z^{2\delta}+\&c.}$$

hæc Figura aſſumendo $s=\frac{n-\delta}{n}$, $x=\frac{1}{s}z^{s}$, & $\lambda=\frac{n-\delta}{\alpha\delta-\beta n}$, migrat in aliam ſibi æqualem cujus Abſciſſa x, ex data Ordinata v, determinatur per æquationem non affectam $\frac{1}{s}v^{\alpha\lambda}\times\overline{e+fv^{n}+gv^{2n}+\&c.}|^{\lambda}\times\overline{k+lv^{n}+mv^{2n}+\&c.}|^{-\lambda}=x$.

COROL. VIII.

Si relatio inter Curvæ alicujus Ordinatam y & Abſciſſam z definitur per æquationem quamvis affectam hujus formæ,

$$y^{\alpha}\times\overline{e+fy^{n}z^{\delta}+gy^{2n}z^{2\delta}+\&c.}=z^{\beta}\times\overline{k+ly^{n}z^{\delta}+my^{2n}z^{2\delta}+\&c.}$$
$$+z^{\gamma}\times\overline{p+qy^{n}z^{\delta}+ry^{2n}z^{2\delta}+\&c.}$$

hæc Figura aſſumendo $s=\frac{n-\delta}{n}$, $x=\frac{1}{s}z^{s}$, $\mu=\frac{\alpha\delta+\beta n}{n-\delta}$, & $\nu=\frac{\alpha\delta+\gamma n}{n-\delta}$, migrat in aliam ſibi æqualem cujus Abſciſſa x ex data Ordinata v determinatur per æquationem minus affectam $v^{\alpha}\times\overline{e+fv^{n}+gv^{2n}+\&c.}=s^{\mu}x^{\mu}\times\overline{k+lv^{n}+mv^{2n}+\&c.}$
$+s^{\nu}x^{\nu}\times\overline{p+qv^{n}+rv^{2n}+\&c.}$

COROL. IX.

Curva omnis cujus Ordinata eſt

$\underline{\pi z^{\theta-1}\times\overline{\nu e+\overline{\nu+n}fz^{n}+\overline{\nu+2n}gz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}}$ in

$\overline{a+b\times\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}|^{\tau}}|^{\omega}$, ſi ſit $\theta=\lambda\nu$, & aſſumantur $x=\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}|^{\pi}$, $\sigma=\frac{\tau}{\pi}$, & $\vartheta=\frac{\lambda-\pi}{\pi}$, migrat in aliam ſibi æqualem cujus Ordinata eſt $x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$. Et nota quod Ordinata prior in hoc Corollario evadit ſimplicior ponendo $\lambda=1$, vel ponendo $\tau=1$, & efficiendo ut radix dignitatis extrahi poſſit cujus index eſt ω, vel etiam ponendo $\omega=-1$, & $\lambda=1=\tau=\sigma=\pi$, ut alios caſus prætercam.

COROL.

COROL. X.

Pro $\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}$ $\overline{\nu ez^{\nu-1}+\overline{\nu+n}fz^{\nu+n-1}+\overline{\nu+2n}gz^{\nu+2n-1}+\&c.}$ $\overline{k+lz^{n}+mz^{2n}+\&c.}$ et $\overline{nlz^{n-1}+2nmz^{2n-1}+\&c.}$ ſcribantur R, $\dot{r}$, S & $\dot{s}$ reſpectivè, & Curva omnis cujus Ordinata eſt $\overline{\pi S\dot{r}+\varphi R\dot{s}}\times R^{\lambda-1}S^{\mu-1}\times\overline{aS^{\upsilon}+bR^{\tau}}|^{\omega}$, ſi ſit $\frac{\mu-\upsilon\omega}{\lambda}=\frac{\upsilon}{\tau}=\frac{\varphi}{\pi}$, $\frac{\tau}{\pi}=\sigma$, $\frac{\lambda-\pi}{\pi}=\vartheta$, & $R^{\pi}S^{\varphi}=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$. Et nota quod Ordinata prior evadit ſimplicior, ponendo unitates pro τ, υ, & λ vel μ, & faciendo ut radix dignitatis extrahi poſſit cujus index eſt ω, vel ponendo $\omega=-1$ vel $\mu=0$.

PROP. X. PROB. III.

Invenire Figuras ſimpliciſſimas cum quibus Curva quævis Geometrice comparari poteſt, cujus Ordinatim applicata y *per æquationem non affectam ex data Abſciſſa* z *determinatur.*

CAS. I.

Sit Ordinata $az^{\vartheta-1}$, & Area erit $\frac{1}{\theta}az^{\theta}$, ut ex Prop. 5. ponendo $b=0=c=d=f=g=h$, & $e=1$, facile colligitur.

CAS. II.

Sit Ordinata $az^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, et ſi Curva cum Figuris rectilineis Geometrice comparari poteſt, quadrabitur per Prop. 5. ponendo $b=0=c=d$. Sin minus convertetur in aliam Curvam ſibi æqualem cujus Ordinata eſt $\frac{a}{n}x^{\frac{\theta-n}{n}}\times\overline{e+fx+gx^{2}+\&c.}|^{\lambda-1}$ per Corol. 2, Prop. 9. Deinde ſi de dignitatum indicibus $\frac{\theta-n}{n}$ & $\lambda-1$ per Prop. 7. rejiciantur unitates

unitates donec dignitates illæ fiant quam minimæ, devenietur ad Figuras simplicissimas quæ hac ratione colligi possunt. Dein harum unaquæque per Corol. 5, Prop. 9. dat aliam quæ nonnunquam simplicior est. Et ex his per Prop. 3. & Corol. 9, & 10, Prop. 9. inter se collatis, Figuræ adhuc simpliciores quandoque prodeunt. Denique ex Figuris simplicissimis assumptis facto regressu computabitur Area quæsita.

C A S. III.

Sit Ordinata $z^{\theta-1} \times \overline{a+bz^{n}+cz^{2n}+\&c.} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, & hæc Figura si quadrari potest, quadrabitur per Prop. 5. Sin minus, distinguenda est Ordinata in partes $z^{\theta-1} \times a \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, $z^{\theta-1} \times bz^{n} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, &c. et per Cas. 2. inveniendæ sunt Figuræ simplicissimæ cum quibus Figuræ partibus illis respondentes comparari possunt.

Nam Areæ figurarum partibus illis respondentium sub signis suis $+$ & $-$ conjunctæ component Aream totam quæsitam.

C A S. IV.

Sit Ordinata $z^{\theta-1} \times \overline{a+bz^{n}+cz^{2n}+\&c.} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$ in $\overline{k+lz^{n}+mz^{2n}+\&c.}|^{\mu-1}$; et si Curva quadrari potest, quadrabitur per Prop. 6. Sin minus, convertetur in simpliciorem per Corol. 4, Prop. 9. ac inde comparabitur cum Figuris simplicissimis per Prop. 8. et Corol. 6, 9 & 10, Prop 9. ut fit in Casu 2 & 3.

C A S. V.

Si Ordinata ex variis partibus constat, partes singulæ pro Ordinatis curvarum totidem habendæ sunt, & Curvæ illæ quotquot quadrari possunt, sigillatim quadrandæ sunt, earumque Ordinatæ de Ordinata tota demendæ.

Dein

Dein Curva quam Ordinatæ pars residua designat seorsim (ut in Casu 2, 3 & 4,) cum Figuris simplicissimis comparanda est cum quibus comparari potest. Et summa Arearum omnium pro Area Curvæ propositæ habenda est.

COROL. I.

Hinc etiam Curva omnis cujus Ordinata est radix quadratica affecta æquationis suæ, cum Figuris simplicissimis seu rectilineis seu curvilineis comparari potest. Nam radix illa ex duabus partibus semper constat quæ seorsim spectatæ non sunt æquationum radices affectæ.

Proponatur æquatio $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$, & extracta radix erit $y = \frac{a^3 + z^3 + a\sqrt{a^4 + 2az^3 - z^4}}{aa + zz}$, cujus pars rationalis $\frac{a^3 + z^3}{aa + zz}$ & pars irrationalis $\frac{a\sqrt{a^4 + 2az^3 - z^4}}{aa + zz}$ sunt Ordinatæ curvarum quæ per hanc Propositionem vel quadrari possunt vel cum Figuris simplicissimis comparari cum quibus collationem Geometricam admittunt.

COROL. II.

Et Curva omnis cujus Ordinata per æquationem quamvis affectam definitur quæ per Corol. 7. Prop. 9. in æquationem non affectam migrat, vel quadratur per hanc Propositionem si quadrari potest, vel comparatur cum Figuris simplicissimis cum quibus comparari potest. Et hac ratione Curva omnis quadratur cujus æquatio est trium terminorum. Nam æquatio illa si affecta sit, transmutatur in non affectam per Corol. 7. Prop. 9. ac deinde per Corol. 2. & 5. Prop. 9. in simplicissimam migrando, dat vel quadraturam Figuræ si quadrari potest, vel Curvam simplicissimam quacum comparatur.

COROL. III.

Et Curva omnis cujus Ordinata per æquationem quamvis affectam definitur quæ per Corol. 8. Prop. 9. in æquationem quadraticam affectam migrat; vel quadratur per hanc Propositionem & hujus Corol. 1. si quadrari potest, vel comparatur cum Figuris simplicissimis cum quibus collationem Geometricam admittit.

SCHOLIUM.

Ubi quadrandæ ſunt Figuræ; ad Regulas haſce generales ſemper recurrere nimis moleſtum eſſet: præſtat Figuras quæ ſimpliciores ſunt & magis uſui eſſe poſſunt ſemel quadrare & quadraturas in Tabulam referre, deinde Tabulam conſulere quoties ejuſmodi Curvam aliquam quadrare oportet. Hujus autem generis ſunt Tabulæ duæ ſequentes, in quibus z denotat Abſciſſam, y Ordinatam rectangulam, & t Aream Curvæ quadrandæ, & d, e, f, g, h, n ſunt quantitates datæ cum ſignis ſuis $+$ & $-$.

TABULA

CURVARUM SIMPLICIORUM QUÆ QUADRARI POSSUNT

	Curvarum Formæ	Curvarum Areæ
I	$dz^{n-1} = y$	$\frac{d}{n}z^{n} = t$
II	$\frac{dz^{n-1}}{e^2+2efz^n+f^2z^{2n}} = y$	$\frac{dz^n}{ne^2+nefz^n} = t$, vel $\frac{-d}{nef+nf^2z^n} = t$
III. 1	$dz^{n-1}\sqrt{e+fz^n} = y$	$\frac{2d}{3nf}R^3 = t$, exiſtente $R = \sqrt{e+fz^n}$
III. 2	$dz^{2n-1}\sqrt{e+fz^n} = y$	$\frac{-4e+6fz^n}{15nf^2}dR^3 = t$
III. 3	$dz^{3n-1}\sqrt{e+fz^n} = y$	$\frac{16e^2-24efz^n+30f^2z^{2n}}{105nf^3}dR^3 = t$
III. 4	$dz^{4n-1}\sqrt{e+fz^n} = y$	$\frac{-96e^3+144e^2fz^n-180ef^2z^{2n}+210f^3z^{3n}}{945nf^4}dR^3 = t$
IV. 1	$\frac{dz^{n-1}}{\sqrt{e+fz^n}} = y$	$\frac{2d}{nf}R = t$
IV. 2	$\frac{dz^{2n-1}}{\sqrt{e+fz^n}} = y$	$\frac{-4e+2fz^n}{3nf^2}dR = t$
IV. 3	$\frac{dz^{3n-1}}{\sqrt{e+fz^n}} = y$	$\frac{16e^2-8efz^n+6f^2z^{2n}}{15nf^3}dR = t$
IV. 4	$\frac{dz^{4n-1}}{\sqrt{e+fz^n}} = y$	$\frac{-96e^3+48e^2fz^n-36ef^2z^{2n}+30f^3z^{3n}}{105nf^4}dR = t$

John Senex ſculpt.

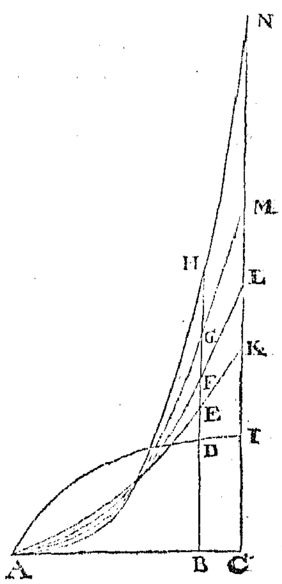
In Tabulis hiſce, ſeries Curvarum cujuſque formæ utrinque in infinitum continuari poteſt. Scilicet in Tabula prima, in numeratoribus Arearum formæ tertiæ & quartæ, numeri coefficientes initialium terminorum (2, —4, 16, —96, 868, &c.) generantur multiplicando numeros — 2, —4, —6, —8, —10, &c. in ſe continuo, & ſubſequentium terminorum coefficientes ex initialibus derivantur multiplicando ipſos gradatim, in Forma quidem tertia, per $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, $-\frac{9}{8}$, $-\frac{11}{10}$, &c. in quarta vero per $-\frac{1}{2}$ $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, $-\frac{9}{10}$, &c. Et Denominatorum coefficientes 3, 15, 105, &c. prodeunt multiplicando numeros 1, 3, 5, 7, 9, &c. in ſe continue.

In ſecunda vero Tabula, ſeries Curvarum formæ primæ, ſecundæ, quintæ, ſextæ, nonæ, & decimæ ope ſolius diviſionis, & formæ reliquæ ope Propoſitionis tertiæ & quartæ, utrinque producuntur in infinitum.

Quinetiam hæ ſeries mutando ſignum numeri n variari ſolent. Sic enim *e. g.* Curva $\frac{d}{z}\sqrt{e+fz^n} = y$, evadit $\frac{d}{z^{\frac{1}{2}n+1}}\sqrt{f+ez^n} = y$.

PROP. XI. THEOR. VIII.

Sit ADIC Curva quævis Abſciſſam habens AB $= z$, & Ordinatam BD $= y$, & ſit AEKC Curva alia cujus Ordinata BE æqualis eſt prioris areæ ADB ad unitatem applicatæ, & AFLC Curva tertia cujus Ordinata BF æqualis eſt ſecundæ Areæ AEB ad unitatem applicatæ, & AGMC Curva quarta cujus Ordinata BG æqualis eſt tertiæ areæ AFB ad unitatem applicatæ, & AHNC Curva quinta cujus Ordinata BH æqualis eſt quartæ Areæ AGB ad unitatem applicatæ, & ſic deinceps in infinitum. Et ſunto A, B, C, D, E, &c. Areæ Curvarum Ordinatas habentium y, zy, z^2y, z^3y, z^4y, &c. et Abſciſſam communem z.



Detur Abſciſſa quævis AC $= t$, ſitque BC $= t - z = x$, & ſunto P, Q, R, S, T Areæ Curvarum Ordinatas habentium y, xy, x^2y, x^3y, x^4y &c. et Abſciſſam communem x.

Terminentur autem hæ Areæ omnes ad Abſciſſam totam datam AC, nec non ad Ordinatam poſitione datam & infinite productam CI:.

Et erit Arearum ſub initio poſitarum

Prima ADIC $= A = P$,

Secunda AEKC $= tA - B = Q$,

Tertia AFLC $= \frac{t^2A - 2tB + C}{2} = \frac{1}{2}R$.

Quarta AGMC $= \frac{t^3A - 3t^2B + 3tC - D}{6} = \frac{1}{6}S$.

Quinta AHNC $= \frac{t^4A - 4t^3B + 6t^2C - 4tD + E}{24} = \frac{1}{24}T$.

COROL.

COROL.

Unde ſi Curvæ quarum Ordinatæ ſunt y, zy, z^2y, z^3y, &c. vel y, xy, x^2y, x^3y, &c. quadrari poſſunt, quadrabuntur etiam Curvæ ADIC, AEKC, AFLC, AGMC, &c. et habebuntur Ordinatæ BE, BF, BG, BH, Areis Curvarum proportionales.

SCHOLIUM.

Quantitatum fluentium Fluxiones eſſe primas, ſecundas, tertias, quartas, aliaſque diximus ſupra. Hæ Fluxiones ſunt ut termini ſerierum infinitarum convergentium.

Ut ſi z^n ſit quantitas fluens & fluendo evadat $\overline{z+o}|^n$, deinde reſolvatur in ſeriem convergentem $z^n + noz^{n-1} + \frac{nn-n}{2}ooz^{n-2} + \frac{n^3-3nn+2n}{6}o^3z^{n-3} +$ &c. terminus primus hujus ſeriei z^n erit quantitas illa fluens, ſecundus noz^{n-1} erit ejus incrementum primum ſeu differentia prima cui naſcenti proportionalis eſt ejus Fluxio prima, tertius $\frac{nn-n}{2}ooz^{n-2}$ erit ut ejus incrementum ſecundum ſeu differentia ſecunda cui naſcenti proportionalis eſt ejus Fluxio ſecunda, quartus $\frac{n^3-3nn+2n}{6}o^3z^{n-3}$ erit ut ejus incrementum tertium ſeu differentia tertia cui naſcenti Fluxio tertia proportionalis eſt, & ſic deinceps in infinitum.

Exponi autem poſſunt hæ Fluxiones per Curvarum Ordinatas BD, BE, BF, BG, BH, &c.

Ut ſi Ordinata BE ($= \frac{ADB}{1}$) ſit quantitas fluens, erit ejus Fluxio prima ut Ordinata BD.

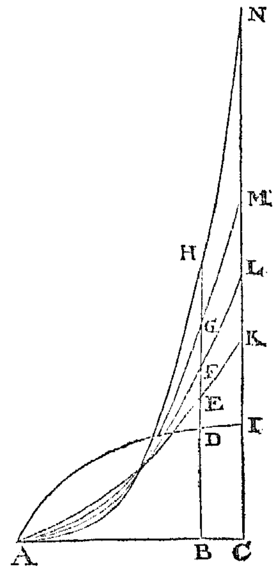
Si BF ($= \frac{AEB}{1}$) ſit quantitas fluens, erit ejus Fluxio prima ut Ordinata BE, & Fluxio ſecunda ut Ordinata BD.

Si BH ($= \frac{AGB}{1}$) ſit quantitas fluens, erunt ejus Fluxiones prima, ſecunda, tertia & quarta, ut Ordinatæ BG, BF, BE, BD reſpective.

Et hinc in æquationibus quæ quantitates tantum duas incognitas involvunt, quarum una eſt quantitas uniformiter fluens & altera eſt Fluxio quælibet quantitatis alterius fluentis, inveniri poteſt fluens illa altera per quadraturam Curvarum. Exponatur enim Fluxio ejus per Ordinatam BD, & ſi hæc ſit Fluxio prima, quæratur Area $ADB = BE \times 1$, ſi Fluxio ſecunda, quæratur Area $AEB = BF \times 1$, ſi Fluxio tertia, quæratur Area $AFB = BG \times 1$, &c. et Area inventa erit exponens fluentis quæſitæ.

Sed

Sed & in æquationibus quæ fluentem & ejus fluxionem primam sine altera fluente, vel duas ejusdem fluentis fluxiones, primam & secundam, vel secundam & tertiam, vel tertiam & quartam, &c. sine alterutra fluente involvunt: inveniri possunt fluentes per quadraturam Curvarum. Sit æquatio $aa\dot{v} = av + vv$, existente $v = \text{BE}$, $\dot{v} = \text{BD}$, $z = \text{AB}$, & $\dot{z} = 1$, & æquatio illa complendo dimensiones Fluxionum, evadet $aa\dot{v} = av\dot{z} + vv\dot{z}$, seu $\frac{aa\dot{v}}{av+vv} = \dot{z}$.



Jam fluat v uniformiter, & sit ejus Fluxio $\dot{v} = 1$, & erit $\frac{aa}{av+vv} = \dot{z}$, & quadrando Curvam cujus Ordinata est $\frac{aa}{av+vv}$ & Abscissa v, habebitur fluens z. Adhæc sit æquatio $aa\ddot{v} = a\dot{v} + \dot{v}\dot{v}$, existente $v = \text{BF}$, $\dot{v} = \text{BE}$, $\ddot{v} = \text{BD}$, & $z = \text{AB}$, & per relationem inter $\ddot{v}$ & $\dot{v}$ seu BD & BE invenietur relatio inter AB & BE, ut in exemplo superiore. Deinde per hanc relationem invenietur relatio inter AB & BF quadrando Curvam AEB.

Æquationes quæ tres incognitas quantitates involvunt aliquando reduci possunt ad æquationes quæ duas tantum involvunt, & in his casibus fluentes invenientur ex fluxionibus ut supra. Sit æquatio $a - bx^m = cxy^n\dot{y} + dy^{2n}\dot{y}\dot{y}$, ponatur $y^n\dot{y} = \dot{v}$, & erit $a - bx^m = cx\dot{v} + d\dot{v}\dot{v}$. Hæc æquatio quadrando Curvam cujus Abscissa est x & Ordinata $\dot{v}$ dat Aream v; & æquatio altera $y^n\dot{y} = \dot{v}$, regrediendo ad fluentes, dat $\frac{1}{n+1}y^{n+1} = v$: Unde habetur fluens y.

Quinetiam in æquationibus, quæ tres incognitas involvunt, & ad æquationes quæ duas tantum involvunt, reduci non possunt, fluentes quandoque prodeunt per quadraturam Curvarum. Sit æquatio $\overline{ax^m + bx^n}|^p = rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$, existente $\dot{x} = 1$; Et pars posterior $rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$, regrediendo ad fluentes, fit $ex^ry^s - \frac{f}{t+1}y^{t+1}$, quæ proinde est ut Area Curvæ cujus Abscissa est x & Ordinata $\overline{ax^m + bx^n}|^p$, & inde datur fluens y.

Sit æquatio $\dot{x} \times \overline{ax^m + bx^n}|^p = \frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$, Et fluens cujus Fluxio est $\dot{x} \times \overline{ax^m + bx^n}|^p$ erit ut Area Curvæ cujus Abscissa est x & Ordinata est $\overline{ax^m + bx^n}|^p$. Item fluens cujus Fluxio est $\frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$ erit ut Area Curvæ cujus Abscissa est y & Ordinata $\frac{dy^{n-1}}{\sqrt{e+fy^n}}$, id est (per Casum 1. Formæ quartæ Tab. I.) ut Area $\frac{2d}{nf}\sqrt{e+fy^n}$. Pono ergo $\frac{2d}{nf}\sqrt{e+fy^n}$ æqualem Areæ Curvæ cujus Abscissa est x & Ordinata $\overline{ax^m + bx^n}|^p$, & habebitur fluens y.

Et nota quod fluens omnis, quæ ex Fluxione prima colligitur, augeri potest vel minui quantitate quavis non fluente. Quæ ex Fluxione secunda colligitur, augeri potest vel minui quantitate quavis cujus Fluxio secunda nulla est. Quæ ex fluxione tertia colligitur, augeri potest vel minui quantitate quavis cujus fluxio tertia nulla est. Et sic deinceps in infinitum.

Postquam fluentes ex fluxionibus collectæ sunt, si de veritate Conclusionis dubitatur, fluxiones fluentium inventarum vicissim colligendæ sunt, & cum Fluxionibus sub initio propositis comparandæ. Nam si prodeunt æquales, Conclusio recte se habet: sin minus, corrigendæ sunt fluentes sic, ut earum Fluxiones fluxionibus sub initio propositis æquentur. Nam & fluens pro lubitu assumi potest, & assumptio corrigi, ponendo fluxionem fluentis assumptæ æqualem Fluxioni propositæ, & terminos homologos inter se comparando.

Et his principiis via ad majora sternitur.

ENUMERATIO
LINEARUM
TERTII ORDINIS.

T A B…

CURVARUM SIMPLICIORUM QUÆ CU…

Sit jam aGD vel PGD vel GDS Sectio Conica cujus Area ad Qua…
Semiaxis conjugatus AP, datum Abſciſsæ principium A vel a vel α…
vel aBDG vel αBDG = σ, exiſtente αG Ordinata ad punctum α. Junga…
parallelogrammum ABDO. Et ſiquando ad quadraturam Curvæ propoſitæ…
Ordinata v, et Area σ. Sit autem ÷ differentia duarum quantitatum ubi…
Et in Forma ſexta ſcribatur p pro $\sqrt{ff - 4eg}$.

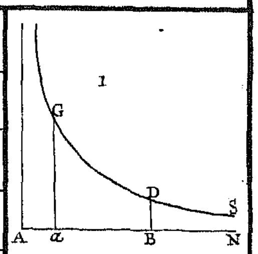
CURVARUM FORMÆ			SECTIONIS CONICÆ: Abſciſsa	SECTIONIS CONICÆ: Ordinata
I.	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fx} = v$
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fx} = v$
	3	$\frac{dz^{3\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fx} = v$
II.	1	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}} = y$	$\sqrt{\frac{d}{e+fz^{\eta}}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f}x^2} = v$
	2	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}} = y$	$\sqrt{\frac{d}{e+fz^{\eta}}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f}x^2} = v$
	3	$\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}} = y$	$\sqrt{\frac{d}{e+fz^{\eta}}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f}x^2} = v$
III.	1	$\frac{d}{z}\sqrt{e+fz^{\eta}} = y$	$\frac{1}{z^{\eta}} = x^2$	$\sqrt{f + ex^2} = v$
		vel sic	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$
	2	$\frac{d}{z^{\eta+1}}\sqrt{e+fz^{\eta}} = y$	$\frac{1}{z^{\eta}} = x^2$	$\sqrt{f + ex^2} = v$
		vel sic	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$
	3	$\frac{d}{z^{2\eta+1}}\sqrt{e+fz^{\eta}} = y$	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$
	4	$\frac{d}{z^{3\eta+1}}\sqrt{e+fz^{\eta}} = y$	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$
IV.	1	$\frac{d}{z\sqrt{e+fz^{\eta}}} = y$	$\frac{1}{z^{\eta}} = x^2$	$\sqrt{f + ex^2} = v$
		vel sic	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$
	2	$\frac{d}{z^{\eta+1}\sqrt{e+fz^{\eta}}} = y$	$\frac{1}{z^{\eta}} = x^2$	$\sqrt{f + ex^2} = v$
		vel sic	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$
	3	$\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}} = y$	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$
	4	$\frac{d}{z^{3\eta+1}\sqrt{e+fz^{\eta}}} = y$	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx + ex^2} = v$

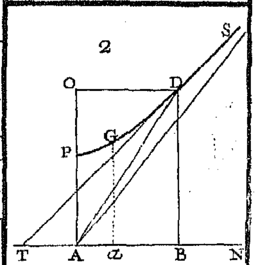
U L A

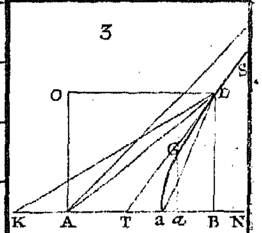
EILIPSI ET HYPERBOLA COMPARARI POSSUNT.

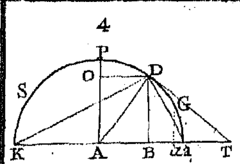
aturam Curvæ propoſitæ requiritur, ſitq; ejus Centrum A, Axis Ka, Vertex a,
bſciſsa AB vel aB vel αB $= x$, Ordinata rectangula BD $= v$, et Area ABDP
ır KD, AD, aD; ducatur Tangens DT occurrens Abſciſsæ AB in T, & compleatur
uiruntur Areæ duarum Sectionum Conicarum, dicatur poſterioris Abſciſsa ξ,
certum eſt utrum poſterior de priori an prior de poſteriori ſubduci debeat.

	CURVARUM AREÆ
	$\frac{1}{\eta}s = t = \frac{\alpha GDB}{\eta}$. *Fig.* 1.
	$\frac{d}{\eta f} z^{\eta} - \frac{e}{\eta f} s = t$.
	$\frac{d}{2\eta f} z^{2\eta} - \frac{de}{\eta f^2} z^{\eta} + \frac{e^2}{\eta f^2} s = t$.
	$\frac{2xv \div 4s}{\eta} = t = \frac{4}{\eta}$ADGa . *Fig.* 3. 4.
	$\frac{2d}{\eta f} z^{\frac{1}{2}\eta} + \frac{4es - 2exv}{\eta f} = t$.
	$\frac{2d}{3\eta f} z^{\frac{3}{2}\eta} - \frac{2de}{\eta f^2} z^{\frac{1}{2}\eta} + \frac{2e^2xv - 4e^2s}{\eta f^2} = t$.
	$\frac{4de}{\eta f} \times \overline{\frac{v^3}{2ex} - s} = t = \frac{4de}{\eta f}$ in aGDT, *vel in* APDB $\div$ TDB *Fig.* 2. 3. 4.
	$\frac{8de^2}{\eta f^2} \times \overline{s - \frac{1}{2}xv - \frac{fv}{4e} + \frac{f^2v}{4e^2x}} = t = \frac{8de^2}{\eta f^2}$ in aGDA $+ \frac{f^2v}{4e^2x}$. *Fig.* 3. 4.
	$-\frac{2d}{\eta} s = t = \frac{2d}{\eta}$APDB ſeu $\frac{2d}{\eta}$ aGDB . *Fig.* 2. 3. 4.
	$\frac{4de}{\eta f} \times \overline{s - \frac{1}{2}xv - \frac{fv}{2e}} = t = \frac{4de}{\eta f} \times$ aGDK . *Fig.* 3. 4.
	$-\frac{d}{\eta} s = t = \frac{d}{\eta} \times -$ aGDB *vel* BDPK . *Fig.* 4.
	$\frac{3dfs - 2dv^3}{6\eta e} = t$:
	$\frac{4d}{\eta f} \times \overline{\frac{1}{2}xv \div s} = t = \frac{4d}{\eta f}$ in PAD *vel in* aGDA. *Fig.* 2. 3. 4.
	$\frac{8de}{\eta f^2} \times \overline{s - \frac{1}{2}xv - \frac{fv}{4e}} = t = \frac{8de}{\eta f^2}$ in aGDA. *Fig.* 3. 4.
	$\frac{2d}{\eta e} \times \overline{s - xv} = t = \frac{2d}{\eta e}$ in POD *vel in* AODGa . *Fig.* 2. 3. 4.
	$\frac{4d}{\eta f} \times \overline{\frac{1}{2}xv \div s} = t = \frac{4d}{\eta f}$ in aDGa *Fig.* 3. 4.
	$\frac{d}{\eta e} \times \overline{3s \div 2xv} = t = \frac{d}{\eta e}$ in 3aDGa $\div \Delta$ aDB . *Fig.* 3. 4.
	$\frac{10dfxv - 15dfs - 2dex^2v}{6\eta e^2} = t$









Iohan. Senex sculp.

RESIDUUM TABULÆ CURVARUM SIMPLICIORU[M]

		Curvarum Formæ	Sectionis Conicæ: Abſciſsa	Sectionis Conicæ: Ordinata
V	1	$\frac{dz^{n-1}}{e+fz^{n}+gz^{2n}} = y$	$\sqrt{\frac{d}{e+fz^{n}+gz^{2n}}} = x$	$\sqrt{\frac{d}{g}+\frac{f^{2}-4eg}{4g^{2}}x^{2}} = v$
		vel sic	$\sqrt{\frac{dz^{2n}}{e+fz^{n}+gz^{2n}}} = x$	$\sqrt{\frac{d}{e}+\frac{f^{2}-4eg}{4e^{2}}x^{2}} = v$
	2	$\frac{dz^{2n-1}}{e+fz^{n}+gz^{2n}} = y$	$\left\{\begin{array}{l}\sqrt{\frac{d}{e+fz^{n}+gz^{2n}}} = x \\ fz^{n}+gz^{2n} = \xi\end{array}\right.$	$\left.\begin{array}{l}\sqrt{\frac{d}{g}+\frac{f^{2}-4eg}{4g^{2}}x^{2}} = v \\ \frac{1}{e+\xi} = \Upsilon\end{array}\right\}$
VI	1	$\frac{dz^{\frac{1}{2}n-1}}{e+fz^{n}+gz^{2n}} = y$	$\left\{\begin{array}{l}\sqrt{\frac{2dg}{f-p+2gz^{n}}} = x \\ \sqrt{\frac{2dg}{f+p+2gz^{n}}} = \xi\end{array}\right.$	$\left.\begin{array}{l}\sqrt{d+\frac{-f+p}{2g}x^{2}} = v \\ \sqrt{d+\frac{-f-p}{2g}\xi^{2}} = \Upsilon\end{array}\right\}$
	2	$\frac{dz^{\frac{3}{2}n-1}}{e+fz^{n}+gz^{2n}} = y$	$\left\{\begin{array}{l}\sqrt{\frac{2dez^{n}}{fz^{n}-pz^{n}+2e}} = x \\ \sqrt{\frac{2dez^{n}}{fz^{n}+pz^{n}+2e}} = \xi\end{array}\right.$	$\left.\begin{array}{l}\sqrt{d+\frac{-f+p}{2e}x^{2}} = v \\ \sqrt{d+\frac{-f-p}{2e}\xi^{2}} = \Upsilon\end{array}\right\}$
VII	1	$\frac{d}{z}\sqrt{e+fz^{n}+gz^{2n}} = y$	$\left\{\begin{array}{l}z^{n} = x \\ \frac{1}{z^{n}} = \xi\end{array}\right.$	$\left.\begin{array}{l}\sqrt{e+fx+gx^{2}} = v \\ \sqrt{g+f\xi+e\xi^{2}} = \Upsilon\end{array}\right\}$
	2	$dz^{n-1}\sqrt{e+fz^{n}+gz^{2n}} = y$	$z^{n} = x$	$\sqrt{e+fx+gx^{2}} = v$
	3	$dz^{2n-1}\sqrt{e+fz^{n}+gz^{2n}} = y$	$z^{n} = x$	$\sqrt{e+fx+gx^{2}} = v$
	4	$dz^{3n-1}\sqrt{e+fz^{n}+gz^{2n}} = y$	$z^{n} = x$	$\sqrt{e+fx+gx^{2}} = v$
VIII	1	$\frac{dz^{n-1}}{\sqrt{e+fz^{n}+gz^{2n}}} = y$	$z^{n} = x$	$\sqrt{e+fx+gx^{2}} = v$
	2	$\frac{dz^{2n-1}}{\sqrt{e+fz^{n}+gz^{2n}}} = y$	$z^{n} = x$	$\sqrt{e+fx+gx^{2}} = v$
	3	$\frac{dz^{3n-1}}{\sqrt{e+fz^{n}+gz^{2n}}} = y$	$z^{n} = x$	$\sqrt{e+fx+gx^{2}} = v$
	4	$\frac{dz^{4n-1}}{\sqrt{e+fz^{n}+gz^{2n}}} = y$	$z^{n} = x$	$\sqrt{e+fx+gx^{2}} = v$
IX	1	$\frac{dz^{n-1}\sqrt{e+fz^{n}}}{g+hz^{n}} = y$	$\sqrt{\frac{d}{g+hz^{n}}} = x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}} = v$
	2	$\frac{dz^{2n-1}\sqrt{e+fz^{n}}}{g+hz^{n}} = y$	$\sqrt{\frac{d}{g+hz^{n}}} = x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}} = v$
X	1	$\frac{dz^{n-1}}{g+hz^{n}\sqrt{e+fz^{n}}} = y$	$\sqrt{\frac{d}{g+hz^{n}}} = x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}} = v$
	2	$\frac{dz^{2n-1}}{g+hz^{n}\sqrt{e+fz^{n}}} = y$	$\sqrt{\frac{d}{g+hz^{n}}} = x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}} = v$
XI	1	$dz^{-1}\sqrt{\frac{e+fz^{n}}{g+hz^{n}}} = y$	$\left\{\begin{array}{l}\sqrt{g+hz^{n}} = x \\ \sqrt{h+gz^{-n}} = \xi\end{array}\right.$	$\left.\begin{array}{l}\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}} = v \\ \sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^{2}} = \Upsilon\end{array}\right\}$
	2	$dz^{n-1}\sqrt{\frac{e+fz^{n}}{g+hz^{n}}} = y$	$\sqrt{g+hz^{n}} = x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}} = v$
	3	$dz^{2n-1}\sqrt{\frac{e+fz^{n}}{g+hz^{n}}} = y$	$\sqrt{g+hz^{n}} = x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}} = v$

QUÆ CUM ELLIPSI ET HYPERBOLA COMPARARI POSSUNT

CURVARUM AREÆ

$$\frac{xv - 2s}{\eta} = t.$$

$$\frac{2s - xv}{\eta} = t.$$

$$\frac{d\sigma + 2fs - fxv}{2\eta g} = t.$$

$$\frac{2xv - 4s - 2\xi\tau + 4\sigma}{\eta p} = t.$$

$$\frac{4s - 2xv - 4\sigma + 2\xi\tau}{\eta p} = t.$$

$$\frac{4de^2\xi\tau + 2def\tau - 2dfgxv \begin{smallmatrix}+4deg\\-2ffd\end{smallmatrix} v - 8de^2\sigma + 4dfgs}{4\eta eg - \eta ff} = t.$$

$$\frac{d}{\eta}s = t = \frac{d}{\eta} \times \alpha GDB. \quad \textit{Fig. 2.3.4.}$$

$$\frac{d}{3\eta g} v^3 - \frac{df}{2\eta g} s = t.$$

$$\frac{6dgx - 5df}{24\eta g^2} v^3 + \frac{5df^2 - 4deg}{16\eta g^2} s = t.$$

$$\frac{8dgs - 4dgxv - 2dfv}{4\eta eg - \eta f^2} = t = \frac{8dg}{4\eta eg - \eta f^2} \times \alpha GDB \pm \Delta DBA. \quad \textit{Fig. 2.4.}$$

$$-\frac{4dfs + 2dfxv + 4dev}{4\eta eg - \eta f^2} = t.$$

$$\frac{\begin{smallmatrix}3dff\\-4deg\end{smallmatrix} s \begin{smallmatrix}-2dff\\+4deg\end{smallmatrix} xv - 2defv}{4\eta eg^2 - \eta f^2 g} = t.$$

$$\frac{\begin{smallmatrix}36defg\\-15df^3\end{smallmatrix} s \begin{smallmatrix}+8degg\\-2df^2g\end{smallmatrix} x^2v \begin{smallmatrix}+10dfff\\-28defg\end{smallmatrix} xv \begin{smallmatrix}+10deff\\-16de^2g\end{smallmatrix} v}{24\eta eg^3 - 6\eta f^2g^2} = t.$$

$$\frac{\begin{smallmatrix}4fg\\-4eh\end{smallmatrix} s \begin{smallmatrix}-2fg\\+2eh\end{smallmatrix} xv + 2df\frac{v}{x}}{\eta fh} = t.$$

$$\frac{\begin{smallmatrix}4egh\\-4fgg\end{smallmatrix} s \begin{smallmatrix}-2egh\\+2fgg\end{smallmatrix} xv + \frac{2}{3}dh\frac{v^3}{x^3} - 2dfg\frac{v}{x}}{\eta fh^2} = t.$$

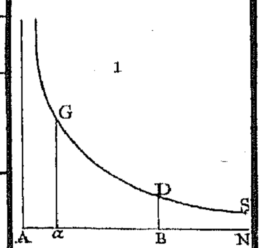
$$\frac{2xv - 4s}{\eta f} = t = \frac{4}{\eta f} ADGa. \quad \textit{Fig. 3.4.}$$

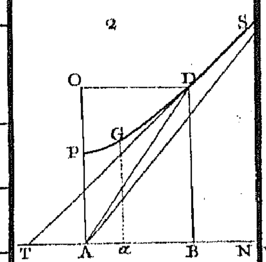
$$\frac{4gs - 2gxv + 2d\frac{v}{x}}{\eta fh} = t.$$

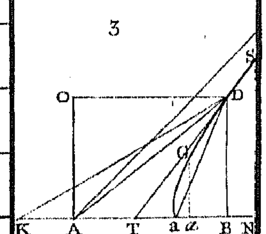
$$\frac{2dxv^3z^{-\eta} - 4dfs - 4de\sigma}{\eta fg - \eta eh} = t.$$

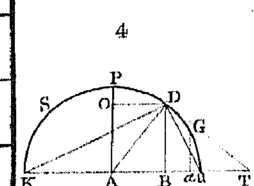
$$\frac{2d}{\eta h}s = t.$$

$$\frac{dhxv^3 \begin{smallmatrix}-3dfg\\-deh\end{smallmatrix} s}{2\eta fh^2} = t.$$









Johan. Senex sculp.

TAB

CURVARUM SIMPLICIORUM QUÆ CU

Sit jam aGD vel PGD vel GDS Sectio Conica cujus Area ad Qu
Semiaxis conjugatus AP, datum Abſciſsæ principium A vel a vel
vel aBDG vel αBDG $= s$, exiſtente αG Ordinata ad punctum α. Jung
parallelogrammum ABDO. Et ſiquando ad quadraturam Curvæ propoſita
Ordinata v, et Area σ. Sit autem $\div$ differentia duarum quantitatum ub
Et in Forma ſexta ſcribatur p pro $\sqrt{ff-4eg}$.

CURVARUM FORMÆ			SECTIONIS CONICÆ	
			Abſciſsa	Ordinata
I.	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	3	$\frac{dz^{3\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
II	1	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	2	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	3	$\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
III	1	$\frac{d}{z}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
IV	1	$\frac{d}{z\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$

CURVARUM AREÆ

$\frac{xv - 2s}{n} = t.$

$\frac{2s - xv}{n} = t.$

$\frac{d\sigma + 2fs - fxv}{2ng} = t.$

$\frac{2xv - 4s - 2\xi\tau + 4\sigma}{np} = t.$

$\frac{4s - 2xv - 4\sigma + 2\xi\tau}{np} = t.$

$\frac{4de^2\xi\tau + 2def\tau - 2dfgxv \,{}^{+4deg}_{-2ff}\, d\,v - 8de^2\sigma + 4dfgs}{4neg - nff} = t.$

$\frac{d}{n}s = t = \frac{d}{n} \times \alpha GDB$. *Fig.* 2. 3. 4.

$\frac{d}{3ng}v^3 - \frac{df}{2ng}s = t.$

$\frac{6dgx - 5df}{24ng^2}v^3 + \frac{5df^2 - 4deg}{16ng^2}s = t.$

$\frac{8dgs - 4dgxv - 2dfv}{4neg - nf^2} = t = \frac{8dg}{4neg - nf^2} \times \alpha GDB \pm \Delta DBA$. *Fig.* 2. 4.

$-\frac{4dfs + 2dfxv + 4dev}{4neg - nf^2} = t.$

$\frac{{}^{3dff}_{-4deg}\, s \,{}^{-2dff}_{+4deg}\, xv - 2defv}{4neg^2 - nf^2g} = t.$

$\frac{{}^{36defg}_{-15df^3}\, s \,{}^{+8degg}_{-2df^2g}\, x^2v \,{}^{+10dfff}_{-28defg}\, xv \,{}^{+10deff}_{-16de^2g}\, v}{24neg^3 - 6nf^2g^2} = t.$

$\frac{{}^{4fg}_{-4eh}\, s \,{}^{-2fg}_{+2eh}\, xv + 2df\frac{v}{x}}{nfh} = t.$

$\frac{{}^{4egh}_{-4fgg}\, s \,{}^{-2egh}_{+2fgg}\, xv + \frac{2}{3}dh\frac{v^3}{x^3} - 2dfg\frac{v}{x}}{nfh^2} = t.$

$\frac{2xv - 4s}{nf} = t = \frac{4}{nf}\, ADGa$. *Fig.* 3. 4.

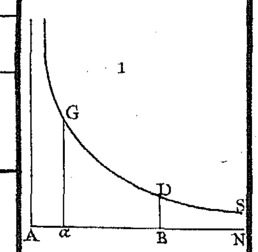
$\frac{4gs - 2gxv + 2d\frac{v}{x}}{nfh} = t.$

$\frac{2dxv^3z^{-n} - 4dfs - 4de\sigma}{nfg - neh} = t.$

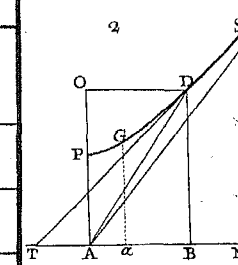
$\frac{2d}{nh}s = t.$

$\frac{dhxv^3 \,{}^{-3dfg}_{-deh}\, s}{2nfh^2} = t.$

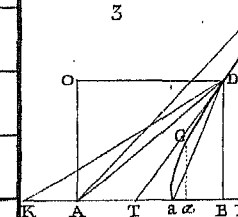
1



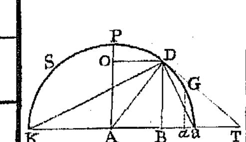
2



3



4



Iohan. Senex sculp.

ENUMERATIO LINEARUM TERTII ORDINIS.

I. *Linearum Ordines.*

Lineæ Geometricæ ſecundum numerum dimenſionum æquationis qua relatio inter Ordinatas & Abſciſſas definitur, vel (quod perinde eſt) ſecundum numerum punctorum in quibus a linea recta ſecari poſſunt, optime diſtinguuntur in Ordines. Qua ratione linea primi Ordinis erit Recta ſola, eæ ſecundi ſive quadratici Ordinis erunt ſectiones Conicæ & Circulus, & eæ tertii ſive cubici Ordinis Parabola Cubica, Parabola *Neiliana*, Ciſſois veterum, & reliquæ quas hic enumerare ſuſcepimus. Curva autem primi Generis, (ſiquidem recta inter Curvas non eſt numeranda) eadem eſt cum Linea ſecundi Ordinis, & Curva ſecundi Generis eadem cum Linea Ordinis tertii. Et Linea Ordinis infiniteſimi ea eſt quam recta in punctis infinitis ſecare poteſt, qualis eſt Spiralis, Cyclois, Quadratrix, & linea omnis quæ per radii vel rotæ revolutiones infinitas generatur.

II. *Proprietates Sectionum Conicarum competunt Curvis superiorum Generum.*

Sectionum Conicarum proprietates præcipuæ a Geometris passim traduntur. Et consimiles sunt proprietates Curvarum secundi Generis, & reliquarum, ut ex sequenti proprietatum præcipuarum enumeratione constabit.

1. *De Curvarum secundi generis Ordinatis, Diametris, Verticibus, Centris, Axibus.*

Si rectæ plures parallelæ & ad Conicam sectionem utrinque terminatæ ducantur, recta duas earum bisecans bisecabit alias omnes, ideoque dicitur *Diameter* figuræ & rectæ bisectæ dicuntur *Ordinatim applicatæ* ad Diametrum, & concursus omnium Diametrorum est *Centrum* figuræ, & intersectio Curvæ & diametri *Vertex* nominatur, & diameter illa *Axis* est cui Ordinatim applicatæ insistunt ad angulos rectos. Et ad eundem modum in Curvis secundi Generis, si rectæ duæ quævis parallelæ ducantur occurrentes Curvæ in tribus punctis: recta quæ ita secat has parallelas ut summa duarum partium ex uno secantis latere ad Curvam terminatarum æquetur parti tertiæ ex altero latere ad curvam terminatæ, eodem modo secabit omnes alias his parallelas curvæque in tribus punctis occurrentes rectas, hoc est, ita ut summa partium duarum ex uno ipsius latere semper æquetur parti tertiæ ex altero latere. Has itaque tres partes quæ hinc inde æquantur, *Ordinatim applicatas*, & rectam secantem cui Ordinatim applicantur *Diametrum*, & intersectionem diametri & Curvæ *Verticem*, & concursum duarum Diametrorum *Centrum* nominare licet. Diameter autem ad Ordinatas rectangula si modo aliqua sit, etiam *Axis* dici potest, & ubi omnes Diametri in eodem puncto concurrunt, istud erit *Centrum generale*.

2. *De Asymptotis & earum proprietatibus.*

Hyperbola primi generis duas *Asymptotos*, ea secundi tres, ea tertii quatuor & non plures habere potest, & sic in reliquis. Et quemadmodum partes lineæ cujusvis rectæ inter Hyperbolam Conicam & duas ejus Asymptotos sunt hinc inde æquales: sic in Hyperbolis secundi Generis si ducatur

ducatur recta quævis secans tam Curvam quam tres ejus Asymptotos in tribus punctis, summa duarum partium istius rectæ quæ a duobus quibusvis Asymptotis in eandem plagam ad duo puncta Curvæ extenduntur, æqualis erit parti tertiæ quæ a tertia Asymptoto in plagam contrariam ad tertium Curvæ punctum extenditur.

3. *De Lateribus rectis & transversis.*

Et quemadmodum in Conicis sectionibus non Parabolicis quadratum Ordinatim applicatæ, hoc est, rectangulum Ordinatarum quæ ad contrarias partes Diametri ducuntur, est ad rectangulum partium Diametri quæ ad Vertices Ellipseos vel Hyperbolæ terminantur, ut data quædam linea quæ dicitur *Latus rectum*, ad partem Diametri quæ inter Vertices jacet & dicitur *Latus transversum* : sic in Curvis non Parabolicis secundi Generis parallelepipedum sub tribus Ordinatim applicatis est ad parallelepipedum sub partibus Diametri ad Ordinatas & tres Vertices figuræ abscissis, in ratione quadam data : in qua ratione si sumantur tres rectæ ad tres partes diametri inter vertices figuræ sitas, singulæ ad singulas, tunc illæ tres rectæ dici possunt *Latera recta* figuræ, & illæ partes Diametri inter Vertices *Latera transversa*. Et sicut in Parabola Conica quæ ad unam & eandem diametrum unicum tantum habet Verticem, rectangulum sub Ordinatis æquatur rectangulo sub parte Diametri quæ ad Ordinatas & Verticem abscinditur & recta quadam data quæ *Latus rectum* dicitur : sic in Curvis secundi Generis quæ non nisi duos habent Vertices ad eandem Diametrum, parallelepipedum sub Ordinatis tribus æquatur parallelepipedo sub duabus partibus Diametri ad Ordinatas & Vertices illos duos abscissis, & recta quadam data quæ proinde *Latus rectum* dici potest.

4. *De Ratione contentorum sub Parallelarum segmentis.*

Denique sicut in Conicis sectionibus ubi duæ parallelæ ad Curvam utrinque terminatæ secantur a duabus parallelis ad Curvam utrinque terminatis, prima a tertia, & secunda a quarta, rectangulum partium primæ est ad rectangulum partium tertiæ ut rectangulum partium secundæ ad rectangulum partium quartæ : sic ubi quatuor tales rectæ occurrunt Curvæ secundi Generis, singulæ in tribus punctis, parallelepipedum partium primæ rectæ erit ad parallelepipedum partium tertiæ, ut parallelepipedum partium secundæ ad parallelepipedum partium quartæ.

5. De

5. *De Cruribus Hyperbolicis & Parabolicis & eorum plagis.*

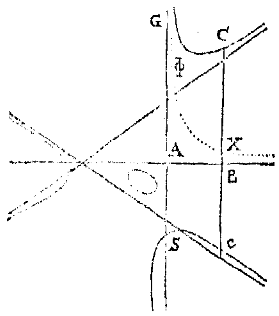
Curvarum secundi & superiorum Generum æque atque primi crura omnia in infinitum progredientia vel *Hyperbolici* sunt generis vel *Parabolici.* Crus *Hyperbolicum* voco quod ad Asymptoton aliquam in infinitum appropinquat; *Parabolicam* quod Asymptoto destituitur. Hæc crura ex Tangentibus optime dignoscuntur. Nam si punctum contactus in infinitum abeat, Tangens cruris Hyperbolici cum Asymptoto coincidet, & Tangens cruris Parabolici in infinitum recedet, evanescet & nullibi reperietur. Invenitur igitur Asymptotos cruris cujusvis quærendo Tangentem cruris illius ad punctum infinite distans. Plaga autem cruris infiniti invenitur quærendo positionem rectæ cujusvis quæ Tangenti parallela est ubi punctum contactus in infinitum abit. Nam hæc recta in eandem plagam cum crure infinito dirigitur.

III. *Reductio Curvarum omnium Generis Secundi ad æquationum casus quatuor.*

Lineæ omnes Ordinis primi, tertii, quinti, septimi & imparis cujusque duo habent ad minimum crura in infinitum versus plagas oppositas progredientia. Et lineæ omnes tertii Ordinis duo habent ejusmodi crura in plagas oppositas progredientia in quas nulla alia earum crura infinita (præterquam in Parabola *Cartesiana)* tendunt.

CAS. I.

Si crura illa sint Hyperbolici generis, sit GAS eorum Asymptotos, & huic

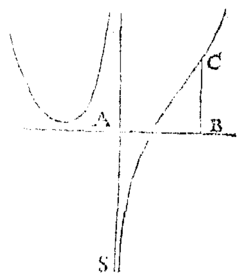


parallela agatur recta quævis CBc ad Curvam utrinque (si fieri potest) terminata, eademque bisecetur in puncto X, & locus puncti illius X erit Hyperbola Conica (puta $X\Phi$) cujus una Asymptotos est AG. Sit ejus altera Asymptotos AB, & æquatio qua relatio inter Ordinatam BC & Abscissam AB definitur, si AB dicatur x, & BC y, semper induet hanc formam $xyy + ey = ax^3 + bxx + cx + d$. Ubi termini, e, a, b, c, d, designant quantitates datas signis suis $+$ & $-$ affectas, quarum quælibet deesse possunt modo ex earum defectu figura in sectionem Conicam non vertatur. Potest autem Hyperbola illa Conica cum Asymptotis suis coincidere, id est punctum X in recta AB locari: & tunc terminus $+ ey$ deest.

CAS. II.

At ſi recta illa CBc non poteſt utrinque ad Curvam terminari, ſed Curvæ in unico tantum puncto occurrit: age quamvis poſitione datam rectam AB Aſymptoto AS occurrentem in A, ut & aliam quamvis BC Aſymptoto illi parallelam Curvæque occurrentem in puncto C, & æquatio qua relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam,

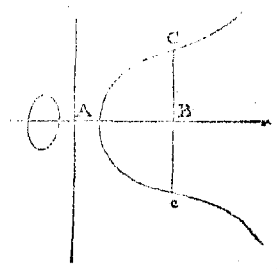
$$xy = ax^3 + bx^2 + cx + d.$$



CAS. III.

Quod ſi crura illa oppoſita Parabolici ſint generis, recta CBc ad Curvam utrinque, ſi fieri poteſt, terminata in plagam crurum ducatur & biſecetur in B, & locus puncti B erit Linea recta. Sit iſta AB, terminata ad datum quodvis punctum A, & æquatio qua relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam,

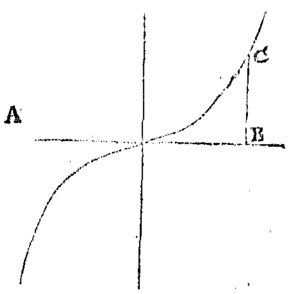
$$yy = ax^3 + bx^2 + cx + d.$$



CAS. IV.

At vero ſi recta illa CBc in unico tantum puncto occurrat Curvæ, ideoque ad Curvam utrinque terminari non poſſit: ſit punctum illud C, & incidat recta illa ad punctum B in rectam quamvis aliam poſitione datam & ad datum quodvis punctum A terminatam AB: & æquatio qua relatio inter Ordinatam BC & Abſciſſam AC definitur, ſemper induet hanc formam,

$$y = ax^3 + bx^2 + cx + d.$$



Nomina formarum.

Enumerando Curvas horum casuum, Hyperbolam vocabimus, *Inscriptam*, quæ tota jacet in Asymptoton angulo ad instar Hyperbolæ conicæ; *Circumscriptam*, quæ Asymptotos secat & partes Abscissas in sinu suo amplectitur; *Ambigenam*, quæ uno crure infinito inscribitur & altero circumscribitur; *Convergentem*, cujus crura concavitate sua seinvicem respiciunt & in plagam eandem diriguntur; *Divergentem*, cujus crura convexitate sua seinvicem respiciunt & in plagas contrarias diriguntur; *Cruribus Contrariis præditam*, cujus crura in partes contrarias convexa sunt & in plagas contrarias infinita; *Conchoidalem*, quæ vertice concavo & cruribus divergentibus ad Asymptoton applicatur; *Anguineam*, quæ flexibus contrariis Asymptoton secat & utrinque in crura contraria producitur; *Cruciformem*, quæ conjugatam decussat; *Nodatam*, quæ seipsam decussat in orbem redeundo; *Cuspidatam*, cujus partes duæ in angulo contactus concurrunt & ibi terminantur; *Punctatam*, quæ conjugatam habet Ovalem infinite parvam id est punctum; & *Puram*, quæ per impossibilitatem duarum radicum Ovali, Nodo, Cuspide & Puncto conjugato privatur. Eodem sensu Parabolam quoque *convergentem*, *divergentem*, *cruribus contrariis præditam*, *cruciformem*, *nodatam*, *cuspidatam*, *punctatam* & *puram* nominabimus.

In casu primo si terminus ax^3 affirmativus est, Figura erit Hyperbola triplex cum sex cruribus Hyperbolicis quæ juxta tres Asymptotos quarum nullæ sunt parallelæ, in infinitum progrediuntur, binæ juxta unamquamque in plagas contrarias. Et hæ Asymptoti si terminus bx^2 non deest, se mutuo secabunt in tribus punctis triangulum (Ddδ) inter se continentes, sin terminus bx^2 deest, convergent omnes ad idem punctum. In priori casu cape $AD = \frac{-b}{2a}$, & $Ad = A\delta = \frac{b}{2\sqrt{a}}$, ac junge Dd, D$\delta$, & erunt Ad, Dd, D$\delta$ tres Asymptoti. In posteriori duc Ordinatam quamvis BC, Ordinatæ principali AG parallelam, & in ea utrinque producta cape hinc inde BF & Bf sibi mutuo æquales & in ea ratione ad AB quam habet $\sqrt{a}$ ad 1, jungeque AF, Af & erunt AG, AF, Af tres Asymptoti. Hanc Hyperbolam vocamus *Redundantem*, quia numero crurum Hyperbolicorum Sectiones Conicas superat.

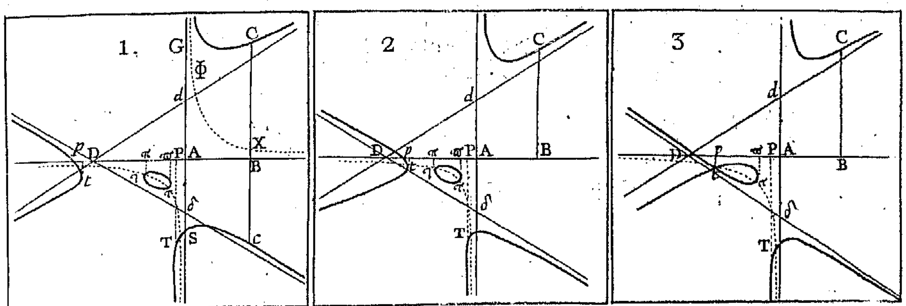
In Hyperbola omni Redundante, si neque terminus ey desit, neque sit $bb - 4ac$ æquale $\pm ae\sqrt{a}$, Curva nullam habebit Diametrum, sin eorum alterutrum accidit Curva habebit unicam Diametrum, & tres si utrumque. Diameter autem semper transit per intersectionem duarum Asymptoton, & bisecat rectas omnes quæ ad Asymptotos illas utrinque terminantur & parallelæ sunt Asymptoto tertiæ. Estque Abscissa AB diameter Figuræ quoties terminus ey deest. *Diametrum* vero absolute dictam hic & in sequentibus in vulgari significatu usurpo, nempe pro Abscissa quæ passim habet Ordinatas binas æquales ad idem punctum hinc inde insistentes.

IV. *Enu-*

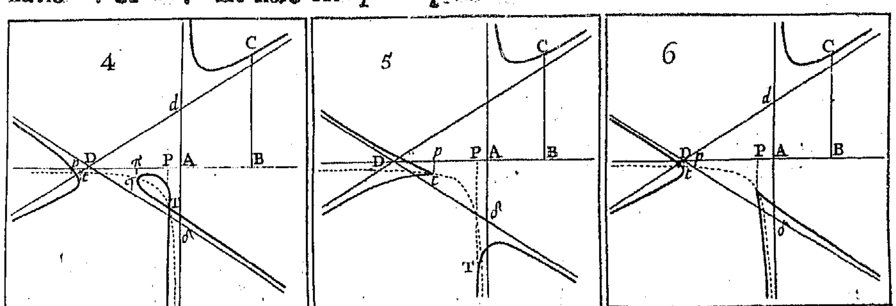
IV. *Enumeratio Curvarum.*

1. *De Hyperbolis novem redundantibus quæ diametro destituuntur & tres habent Asymptotos triangulum capientes.*

Si Hyperbola redundans nullam habet diametrum, quærantur Æquationis hujus $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ radices quatuor seu valores ipsius x. Eæ sunto AP, Aϖ, Aπ, Ap. Erigantur Ordinatæ PT, ϖτ, π7, pt, & hæ tangent Curvam in punctis totidem T, τ, 7, t, & tangendo dabunt limites Curvæ per quos Species ejus innotescet.



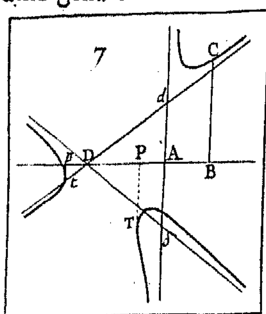
Nam si radices omnes AP, Aϖ, Ap, (*Fig.* 1, 2.) sunt reales, ejusdem signi & inæquales, Curva constat ex tribus Hyperbolis, (inscripta, circumscripta & ambigena) cum *Ovali.* Hyperbolarum una jacet versus D, altera versus d, tertia versus δ, & Ovalis semper jacet intra Triangulum Ddδ, atque etiam inter medios limites 7 & τ, in quibus utique tangitur ab Ordinatis π7 & ϖτ. Et hæc est *Species prima.*

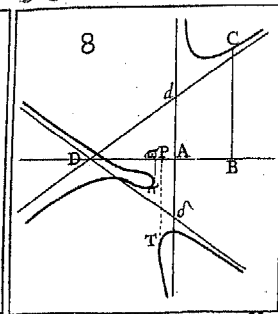


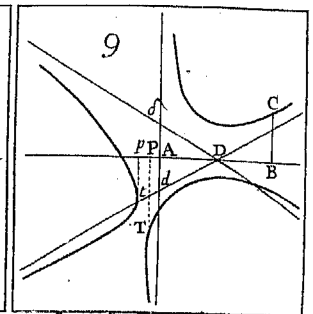
Si e radicibus duæ maximæ Aπ, Ap, (*Fig.* 3.) vel duæ minimæ AP, Aϖ (*Fig.* 4.) æquantur inter se, & ejusdem sunt signi cum alteris duobus, Ovalis & Hyperbola circumscripta sibi invicem junguntur coeuntibus earum punctis contactus 7 & t vel T & τ, & crura Hyperbolæ sese decussando in Ovalem continuantur, figuram *Nodatam* efficientia. Quæ *Species* est *secunda.* Si

Si e radicibus tres maximæ Ap, $A\pi$, $A\varpi$, (*Fig.* 5.) vel tres minimæ $A\pi$, $A\varpi$, AP (*Fig.* 6.) æquentur inter ſe, Nodus in *Cuſpidem* acutiſſimum convertetur. Nam crura duo Hyperbolæ circumſcriptæ ibi in angulo contactus concurrent & non ultra producentur. Et hæc eſt *Species tertia*.

Si e radicibus duæ mediæ $A\varpi$ & $A\pi$ (*Fig.* 7.) æquentur inter ſe, puncta contactus τ & τ coincidunt, & propterea Ovalis interjecta in punctum evanuit, & conſtat figura ex tribus Hyperbolis, inſcripta, circumſcripta & ambigena cum *Puncto* conjugato. Quæ eſt *Species quarta*.

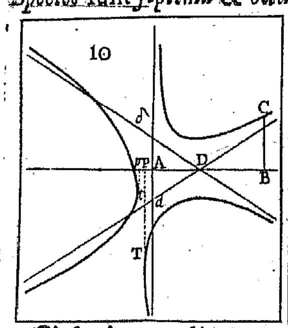


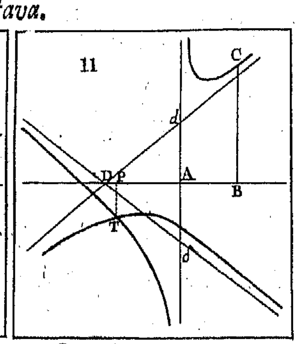


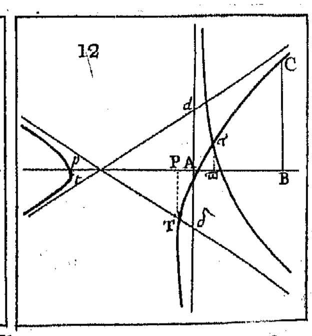


Si duæ ex radicibus ſunt impoſſibiles & reliquæ duæ inæquales & ejuſdem ſigni (nam ſigna contraria habere nequeunt,) *Puræ* habebuntur Hyperbolæ tres ſine Ovali vel Nodo vel Cuſpide vel Puncto conjugato, & hæ Hyperbolæ vel ad latera trianguli ab Aſymptotis comprehenſi vel ad angulos ejus jacebunt; & perinde *Speciem* vel *quintam* (*Fig.* 7. 8.) vel *ſextam* (*Fig.* 9, 10.) conſtituent.

Si e radicibus duæ ſunt æquales & alteræ duæ vel impoſſibiles ſunt (*Fig.* 11. 13.) vel reales (*Fig.* 12. 14.) cum ſignis quæ a ſignis æqualium radicum diverſa ſunt, figura *Cruciformis* habebitur, nempe duæ ex Hyperbolis ſeinvicem decuſſabunt idque vel ad verticem trianguli ab Aſymptotis comprehenſi (*Fig.* 13, 14.) vel ad ejus baſem (*Fig.* 11, 12.) Quæ duæ *Species* ſunt *ſeptima* & *octava*.



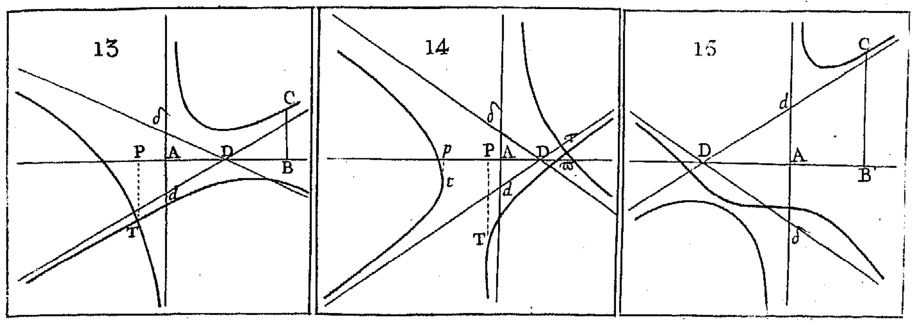




Si denique radices omnes ſunt impoſſibiles (*Fig.* 15.) vel ſi omnes ſunt reales & inæquales (*Fig.* 16.) & earum duæ ſunt affirmativæ & alteræ duæ negativæ, tunc duæ habebuntur Hyperbolæ ad angulos oppoſitos duarum Aſymptoton cum Hyperbola *Anguinea* circa Aſymptoton tertiam. Quæ *Species* eſt *nona*.

Et

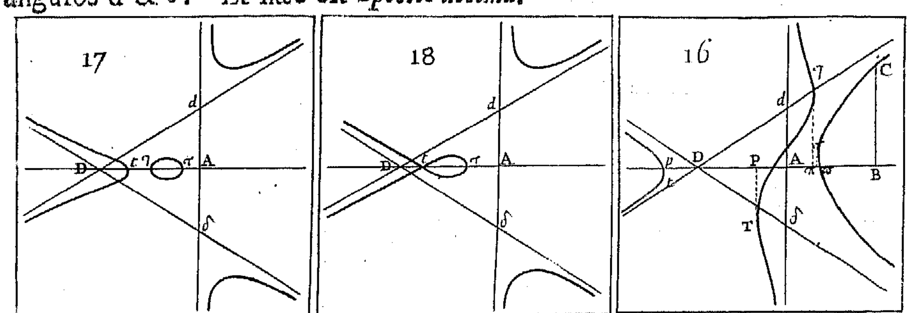
Et hi sunt omnes radicum casus possibiles. Nam si duæ radices sunt æquales inter se, & aliæ duæ sunt etiam inter se æquales, Figura evadet Sectio Conica cum Linea recta.



2. *De Hyperbolis duodecim redundantibus unicam tantum Diametrum habentibus.*

Si Hyperbola redundans habet unicam tantum Diametrum, sit ejus Diameter Abscissa AB, & æquationis hujus $ax^3 + bx^2 + cx + d = 0$ quære tres radices seu valores x.

Si radices illæ sunt omnes reales & ejusdem signi, Figura constabit ex *Ovali* intra triangulum Ddδ (*Fig.* 17.) jacente & tribus Hyperbolis ad angulos ejus, nempe Circumscripta ad angulum D & Inscriptis duabus ad angulos d & δ. Et hæc est *Species decima.*



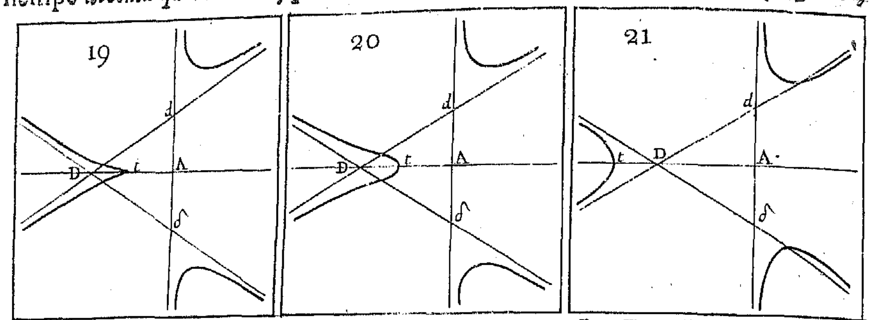
Si radices duæ majores sunt æquales & tertia ejusdem signi, crura Hyperbolæ jacentis versus D (*Fig.* 18.) sese decussabunt in forma *Nodi* propter contactum Ovalis. Quæ *Species* est *undecima.*

Si tres radices sunt æquales, Hyperbola ista fit *Cuspidata* sine Ovali, (*Fig.* 19.) Quæ *Species* est *duodecima.*

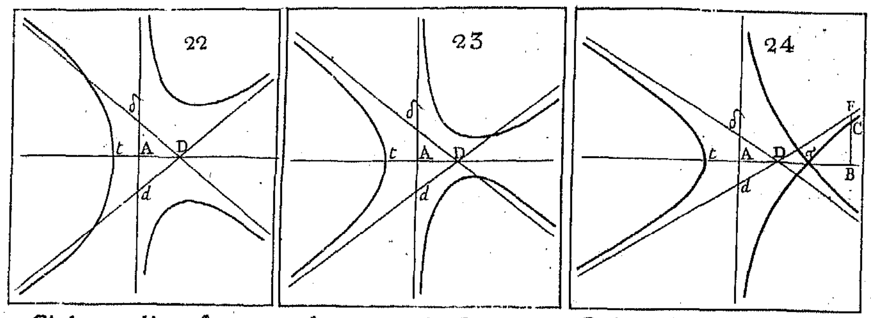
Si radices duæ minores sunt æquales & tertia ejusdem signi, Ovalis in *Punctum* evanuit, (*Fig.* 20.) Quæ *Species* est *decima tertia.*

In speciebus quatuor novissimis Hyperbola quæ jacet versus D, Asymptotos in sinu suo amplectitur, reliquæ duæ in sinu Asymptoton jacent.

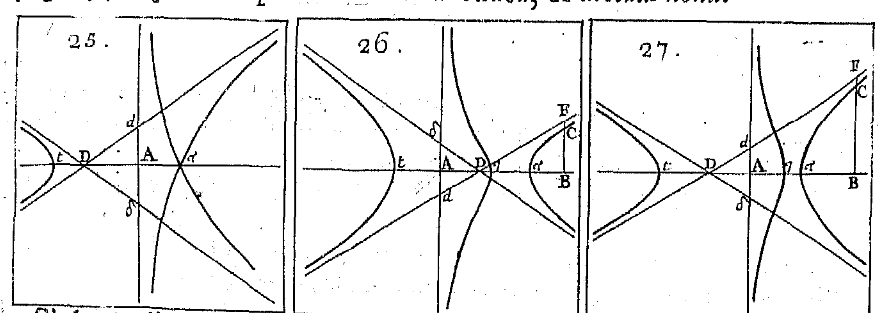
Si duæ ex radicibus ſunt impoſſibiles habebuntur tres Hyperbolæ *Puræ* ſine Ovali decuſſatione vel cuſpide. Et hujus caſus *Species* ſunt quatuor: nempe *decima quarta* ſi Hyperbola Circumſcripta jacet verſus D, (*Fig.* 20.)



& *decima quinta* ſi Hyperbola Inſcripta jacet verſus D, (*Fig.* 21.) *decima ſexta* ſi Hyperbola Circumſcripta jacet ſub baſi dδ trianguli Ddδ, (*Fig.* 22.) & *decima ſeptima* (*Fig.* 23.) ſi Hyperbola inſcripta jacet ſub eadem baſi.



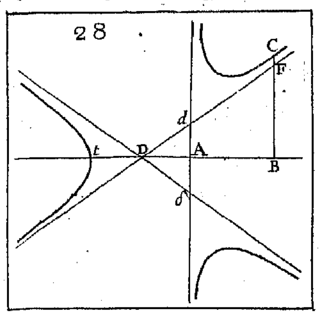
Si duæ radices ſunt æquales & tertia ſigni diverſi figura erit *Cruciformis*. Nempe duæ ex tribus Hyperbolis ſeinvicem decuſſabunt idque vel ad verticem trianguli ab Aſymptotis comprehenſi (*Fig.* 24.) vel ad ejus baſem, (*Fig.* 25.) Quæ duæ *Species* ſunt *decima octava*, & *decima nona*.

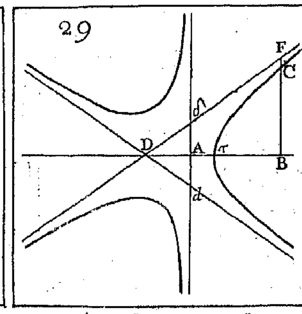


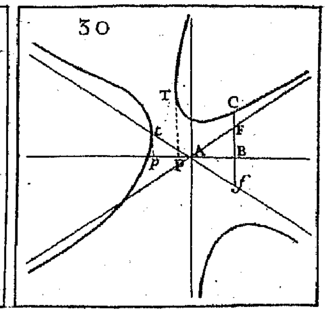
Si duæ radices ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni diverſi, duæ habebuntur Hyperbolæ in oppoſitis angulis duarum Aſymptoton cum

Con-

Conchoidali intermedia. *Conchoidalis* autem vel jacebit ad eaſdem partes Aſymptoti ſuæ cum triangulo ab Aſymptotis conſtituto, (*Fig.* 26.) vel ad partes contrarias, (*Fig.* 27.) Et hi duo caſus conſtituunt *Speciem vigeſimam* & *vigeſimam primam*.

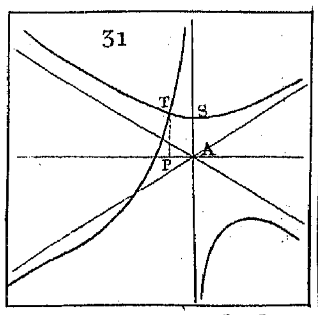


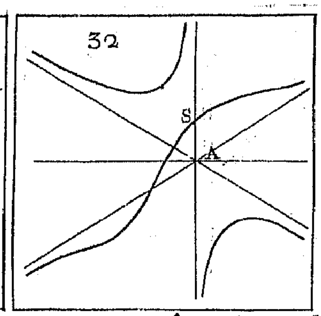


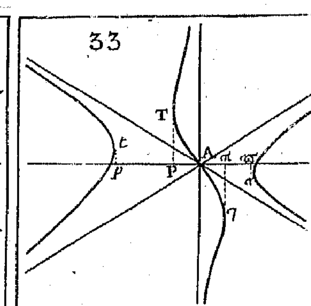


3. *De Hyperbolis duabus redundantibus cum tribus Diametris.*

Hyperbola redundans quæ habet tres diametros, conſtat ex tribus Hyperbolis in ſinubus Aſymptoton jacentibus, idque vel ad angulos trianguli ab Aſymptotis comprehenſi (*Fig.* 28.) vel ad ejus latera (*Fig.* 29.) Caſus prior dat *Speciem vigeſimam ſecundam*, & poſterior *Speciem vigeſimam tertiam*.





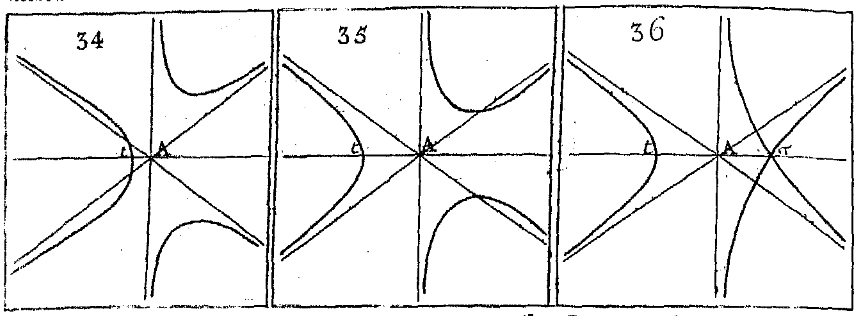


4. *De Hyperbolis novem redundantibus cum Aſymptotis tribus ad commune punctum convergentibus.*

Si tres Aſymptoti in puncto communi ſe mutuo decuſſant, vertuntur ſpecies quinta & ſexta in *vigeſimam quartam*, (*Fig.* 30.) ſeptima & octava in *vigeſimam quintam*, (*Fig.* 31.) & nona in *vigeſimam ſextam* (*Fig.* 32.) ubi Anguinea non tranſit per concurſum Aſymptoton, & in *vigeſimam ſeptimam* ubi tranſit per concurſum illum, (*Fig.* 33.) quo caſu termini *b* ac *d* deſunt, & concurſus Aſymptoton eſt Centrum figuræ ab omnibus ejus partibus oppoſitis æqualiter diſtans. Et hæ quatuor ſpecies diametrum non habent.

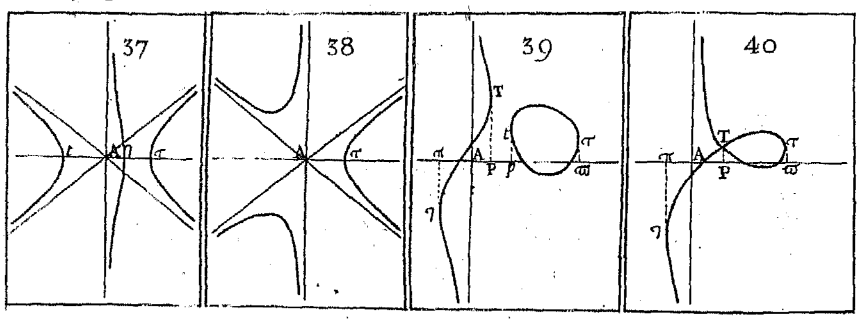
Ver-

Vertuntur etiam *Species* decima quarta ac decima sexta in *vigesimam octavam*, (*Fig.* 34.) decima quinta ac decima septima in *vigesimam nonam*, (*Fig.* 35.) decima octava & decima nona in *tricesimam*, (*Fig.* 36.) & vigesima cum vigesima prima in *tricesimam primam*, (*Fig.* 37.) Et hæ species unicam habent Diametrum.



Ac denique species vigesima secunda & vigesima tertia vertuntur in *Speciem tricesimam secundam* cujus tres sunt Diametri per concursum Asymptoton transeuntes. (*Fig.* 38.)

Quæ omnes conversiones facillime intelliguntur faciendo ut triangulum ab Asymptotis comprehensum diminuatur donec in punctum evanescat.



5. *De Hyperbolis sex defectivis diametrum non habentibus.*

Si in primo æquationum casu terminus ax^3 negativus est, Figura erit Hyperbola defectiva unicam habens Asymptoton & duo tantum crura Hyperbolica juxta Asymptoton illam in plagas contrarias infinite progredientia. Et Asymptotos illa est Ordinata prima & principalis AG. Si terminus ey non deest figura nullam habebit Diametrum, si deest habebit unicam. In priori casu species sic enumerantur.

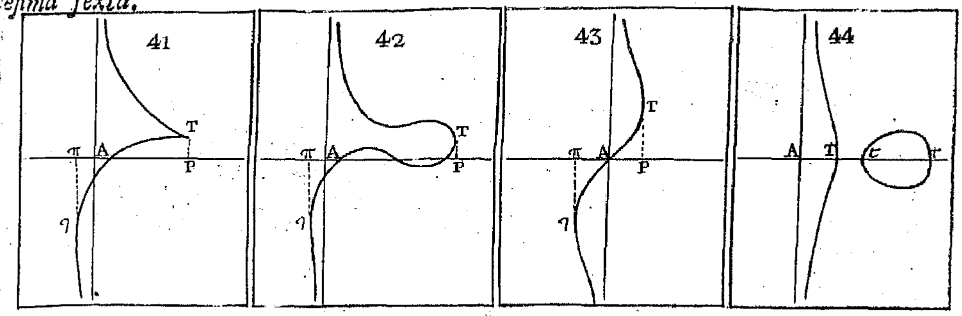
Si æquationis hujus $ax^4 = bx^3 + cx^2 + dx + \frac{1}{4}ee$ radices omnes $A\pi$, AP, Ap, $A\varpi$, (*Fig.* 39.) sunt reales & inæquales, Figura erit Hyperbola Anguinea asymptoton flexu contrario amplexa, cum *Ovali* conjugata. Quæ *Species* est *tricesima tertia*.

Si

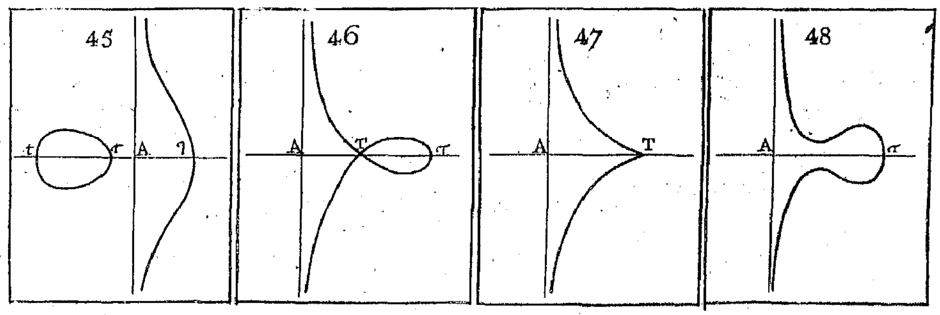
Si radices duæ mediæ AP & Ap (*Fig.* 40) æquentur inter se, Ovalis & Anguinea junguntur sese decussantes in forma *Nodi*. Quæ est *Species tricesima quarta*.

Si tres radices sunt æquales, Nodus vertetur in *Cuspidem* acutissimum in vertice Anguineæ, (*Fig.* 41) Et hæc est *Species tricesima quinta*.

Si e tribus radicibus ejusdem signi duæ maximæ Ap & Aϖ (*Fig.* 43) sibi mutuo æquantur, Ovalis in *Punctum* evanuit. Quæ *Species* est *tricesima sexta*.



Si radices duæ quævis imaginariæ sunt, sola manebit Anguinea *Pura* sine Ovali, Decussatione, Cuspide vel Puncto conjugato. Si Anguinea illa non transit per punctum A (*Fig.* 42) *Species* est *tricesima septima*; sin transit per punctum illud A (id quod contingit ubi termini b ac d desunt,) punctum illud A erit centrum figuræ rectas omnes per ipsum ductas & ad Curvam utrinque terminatas bisecans, (*Fig.* 43.) Et hæc est *Species tricesima octava*.



6. *De Hyperbolis septem defectivis diametrum habentibus.*

In altero casu ubi terminus ey deest & propterea figura Diametrum habet, si æquationis hujus $ax^3 = bx^2 + cx + d$ radices omnes AT, At, Aτ, (*Fig.* 44) sunt reales, inæquales & ejusdem signi, figura erit Hyperbola Conchoidalis cum *Ovali* ad convexitatem. Quæ est *Species tricesima nona*.

Si duæ radices ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni contrarii, *Ovalis* jacebit ad concavitatem Conchoidalis, *Fig.* 45.) Eſtque *Species quadrageſima.*

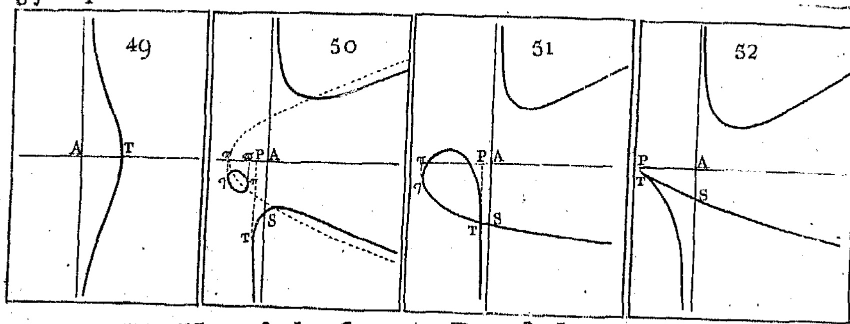
Si radices duæ minores AT, At, (*Fig.* 46) ſunt æquales, & tertia *A*τ eſt ejuſdem ſigni, Ovalis & Conchoidalis jungentur ſeſe decuſſando in modum *Nodi.* Quæ *Species* eſt *quadrageſima prima.*

Si tres radices ſunt æquales, Nodus mutabitur in *Cuſpidem*, & figura erit *Ciſſois Veterum*, (*Fig.* 47.) Et hæc eſt *Species quadrageſima ſecunda.*

Si radices duæ majores ſunt æquales, & tertia eſt ejuſdem ſigni, Conchoidalis habebit *Punctum* conjugatum ad convexitatem ſuam, (*Fig.* 49.) Eſtque *Species quadrageſima tertia.*

Si radices duæ ſunt æquales, & tertia eſt ſigni contrarii Conchoidalis habebit *Punctum* conjugatum ad concavitatem ſuam, (*Fig.* 49.) Eſtque *Species quadrageſima quarta.*

Si radices duæ ſunt impoſſibiles habebitur Conchoidalis *Pura* ſine Ovali, Nodo, Cuſpide vel Puncto conjugato (*Fig.* 48. 49.) Quæ *Species* eſt *quadrageſima quinta.*



7. De Hyperbolis ſeptem Parabolicis Diametrum non habentibus.

Si quando in primo æquationum caſu terminus ax^3 deeſt & terminus bx^2 non deeſt, Figura erit Hyperbola Parabolica duo habens crura Hyperbolica ad unam Aſymptoton SAG & duo Parabolica in plagam unam & eandem convergentia. Si terminus ey non deeſt figura nullam habebit diametrum, ſin deeſt habebit unicam. In priori caſu Species ſunt hæ.

Si tres radices AP, Aϖ, Aπ (*Fig.* 50.) æquationis hujus $bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ ſunt inæquales & ejuſdem ſigni, figura conſtabit ex *Ovali* & aliis duabus Curvis quæ partim Hyperbolicæ ſunt & partim Parabolicæ. Nempe crura Parabolica continuo ductu junguntur cruribus Hyperbolicis ſibi proximis. Et hæc eſt *Species quadrageſima ſexta.*

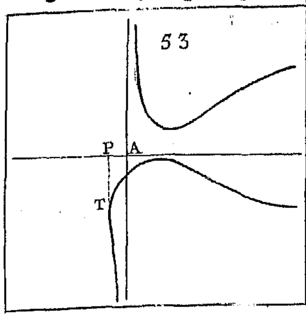
Si radices duæ minores ſunt æquales, & tertia eſt ejuſdem ſigni, Ovalis & una Curvarum illarum Hyperbolo-Parabolicarum junguntur & ſe decuſſant in formam *Nodi* (*Fig.* 51.) Quæ *Species* eſt *quadrageſima ſeptima.*

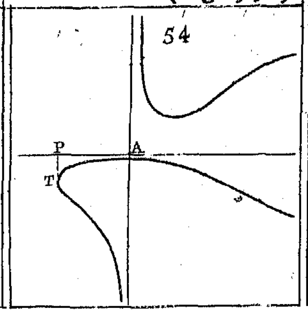
Si

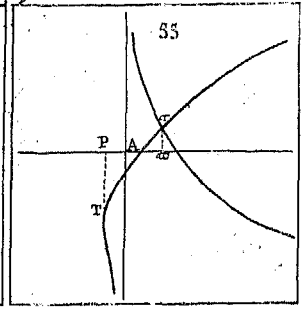
Si tres radices ſunt æquales, Nodus ille in Cuſpidem vertitur (*Fig.* 52.) Eſtque *Species quadrageſima octava.*

Si radices duæ majores ſunt æquales & tertia eſt ejuſdem ſigni, Ovalis in *Punctum* conjugatum evanuit (*Fig.* 53.) Quæ *Species* eſt *quadrageſima nona.*

Si duæ radices ſunt impoſſibiles, manebunt *Puræ* illæ duæ curvæ Hyperbolo-parabolicæ ſine Ovali, Decuſſatione, Cuſpide vel Puncto conjugato; & *Speciem quinquageſimam* conſtituent. (*Fig.* 53. 54.)

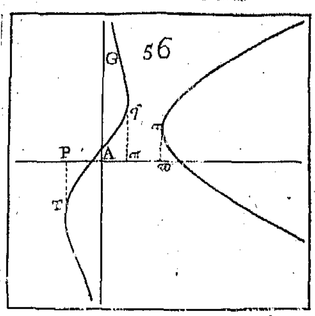


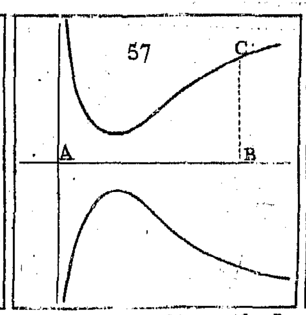


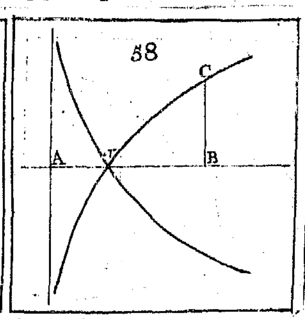


Si radices duæ ſunt æquales & tertia eſt ſigni contrarii, Curvæ illæ Hyperbolo-parabolicæ junguntur ſeſe decuſſando in morem crucis (*Fig.* 55.) Eſtque *Species quinquageſima prima.*

Si radices duæ ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni contrarii, figura evadet Hyperbola Anguinea circa Aſymptoton AG, (*Fig.* 56) cum Parabola conjugata. Et hæc eſt *Species quinquageſima ſecunda.*







8. *De Hyperbolis quatuor Parabolicis Diametrum habentibus.*

In altero caſu ubi terminus ey deeſt & figura Diametrum habet, ſi duæ radices æquationis hujus $bx^2 + cx + d = 0$ ſunt impoſſibiles, duæ habentur figuræ Hyperbolo-parabolicæ a Diametro AB (*Fig.* 57.) hinc inde æqualiter diſtantes. Quæ *Species* eſt *quinquageſima tertia.*

Si æquationis illius radices duæ ſunt impoſſibiles, Figuræ Hyperbolo-parabolicæ junguntur ſeſe decuſſantes in morem crucis; & *Speciem quinquageſimam quartam* conſtituunt. *Fig.* 58.

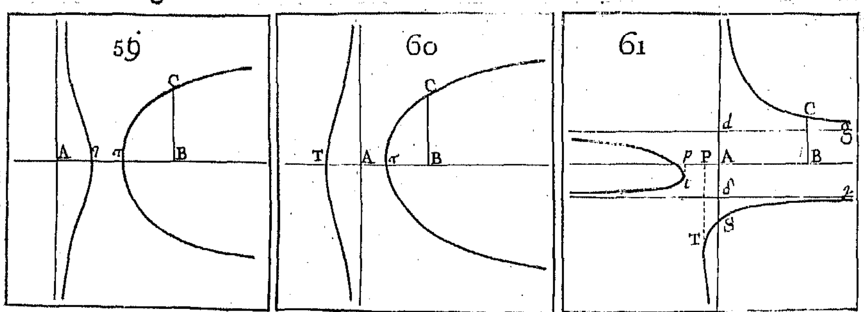
Si

Si radices illæ sunt inæquales & ejusdem signi, habetur Hyperbola Conchoidalis cum Parabola ex eodem latere Asymptoti (*Fig.* 59.) Estque *Species quinquagesima quinta.*

Si radices illæ sunt signi contrarii, habetur Conchoidalis cum Parabola ad alteras partes Asymptoti (*Fig.* 60.) Quæ *Species* est *quinquagesima sexta.*

9. *De Quatuor Hyperbolismis Hyperbolæ.*

Si quando in primo æquationum casu terminus uterque ax^3 & bx^2 deest, figura erit Hyperbolismus sectionis alicujus Conicæ. Hyperbolismum figuræ voco cujus Ordinata prodit applicando contentum sub Ordinata figuræ illius & recta data ad Abscissam communem. Hac ratione linea recta vertitur in Hyperbolam Conicam, & sectio omnis Conica vertitur in aliquam figurarum quas hic Hyperbolismos sectionum Conicarum voco. Nam æquatio ad figuras de quibus agimus, nempe $xy^2 + ey = cx + d$, dat $y = \frac{e \pm \sqrt{ee + 4dx + 4cxx}}{2x}$ quæ generatur applicando contentum sub Ordinata sectionis Conicæ $\frac{e \pm \sqrt{ee + 4dx + 4cxx}}{2m}$ & recta data m, ad curvarum Abscissam communem x. Unde liquet quod figura genita Hyperbolismus erit Hyperbolæ, Ellipseos vel Parabolæ, perinde ut terminus cx affirmativus est vel negativus vel nullus.



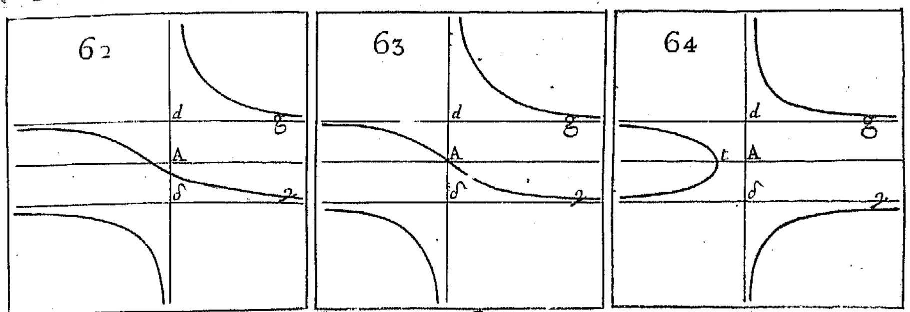
Hyperbolismus Hyperbolæ tres habet Asymptotos, quarum una est Ordinata prima & principalis Ad, alteræ duæ sunt parallelæ Abscissæ AB, ab eadem hinc inde æqualiter distant. In Ordinata principali Ad, cape Ad, Aδ hinc inde æquales quantitati $\sqrt{c}$; & per puncta d ac δ age dg, $\delta\gamma$ Asymptotos Abscissæ AB parallelas.

Ubi terminus ey non deest figura nullam habet diametrum. In hoc casu, si æquationis hujus $cx^2 + dx + \frac{1}{4}ee = 0$ radices duæ AP, Ap (*Fig.* 61) sunt reales & inæquales (nam æquales esse nequeunt nisi figura sit Conica sectio) figura constabit ex tribus Hyperbolis sibi oppositis quarum una jacet inter

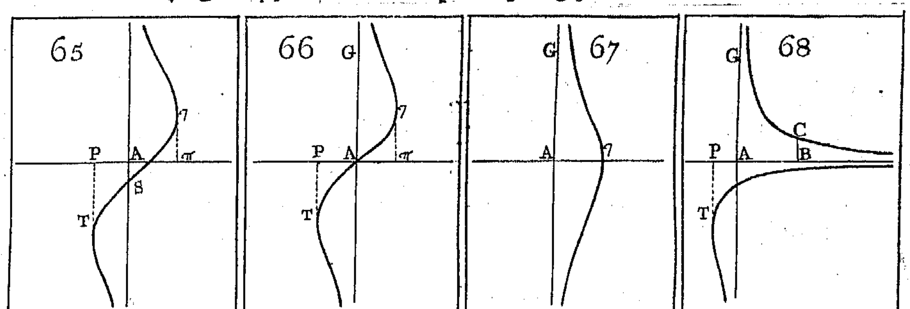
Asymp-

Aſymptotos parallelas & alteræ duæ jacent extra. Et hæc eſt *Species quinquageſima ſeptima.*

Si radices illæ duæ ſunt impoſſibiles, habentur Hyperbolæ duæ oppoſitæ extra Aſymptotos parallelas & Anguinea Hyperbolica intra eaſdem. Hæc figura duarum eſt ſpecierum. Nam centrum non habet ubi terminus d non deeſt (*Fig.* 62;) ſed ſi terminus ille deeſt punctum A eſt ejus centrum (*Fig.* 63.) Prior *Species* eſt *quinquageſima octava*, poſterior *quinquageſima nona.*



Quod ſi terminus ey deeſt, figura conſtabit ex tribus Hyperbolis oppoſitis, quarum una jacet inter Aſymptotos parallelas & alteræ duæ jacent extra ut in ſpecie quinquageſima quarta, & præterea diametrum habet quæ eſt Abſciſſa AB (*Fig.* 64.) Et hæc eſt *Species ſexageſima.*



10. *De tribus Hyperboliſmis Ellipſeos.*

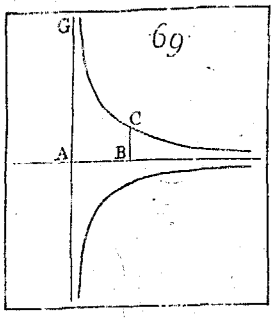
Hyperboliſmus Ellipſeos per hanc æquationem definitur $xy^2 + ey = cx + d$, & unicam habet Aſymptoton quæ eſt Ordinata principalis Ad (*Fig.* 65.) Si terminus ey non deeſt, figura eſt Hyperbola Anguinea ſine diametro, atque etiam ſine centro ſi terminus d non deeſt. Quæ *Species* eſt *ſexageſima prima.*

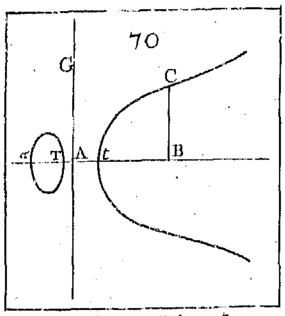
At ſi terminus d deeſt, figura habet centrum ſine diametro, & centrum ejus eſt punctum A (*Fig.* 66.) *Species* vero eſt *ſexageſima ſecunda.*

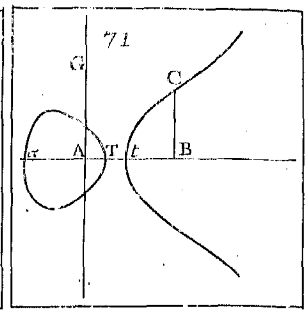
Et ſi terminus ey deeſt, & terminus d non deeſt, figura eſt Conchoidalis ad Aſymptoton AG *Fig.* 67,) habetque diametrum ſine centro, & diameter ejus eſt Abſciſſa AB. Quæ *Species* eſt *ſexageſima tertia.*

11. *De duobus Hyperboliſmis Parabolæ.*

Hyperboliſmus Parabolæ per hanc æquationem definitur $xy^2 + ey = d$; & duas habet Aſymptotos, Abſciſſam AB & Ordinatam primam & principalem AG. Hyperbolæ vero in hac figura ſunt duæ, non in Aſymptoton angulis oppoſitis ſed in angulis qui ſunt deinceps jacentes, idque ad utrumque latus abſciſſæ AB, & vel ſine diametro ſi terminus ey habetur, (*Fig.* 68) vel cum diametro ſi terminus ille deeſt (*Fig.* 69.) Quæ duæ *Species* ſunt *ſexageſima quarta* & *ſexageſima quinta*.

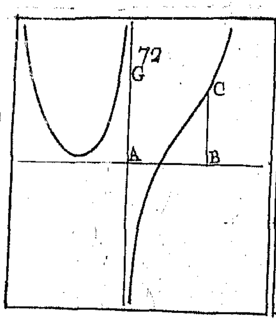


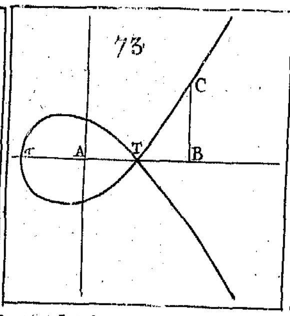


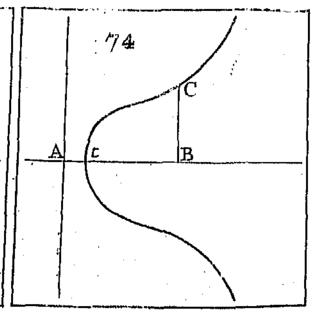


12. *De Tridente.*

In Secundo æquationum caſu habebatur æquatio $xy = ax^3 + bx^2 + cx + d$. Et figura in hoc caſu habet quatuor crura infinita quorum duo ſunt Hyperbolica circa Aſymptoton AG (*Fig.* 72) in contrarias partes tendentia & duo Parabolica convergentia & cum prioribus ſpeciem Tridentis fere efformantia. Eſtque hæc Figura Parabola illa per quam *Carteſius* æquationes ſex dimenſionum conſtruxit. Hæc eſt igitur *Species ſexageſima ſexta*.







13. *De Parabolis quinque divergentibus.*

In Tertio caſu æquatio erat $y^2 = ax^3 + bx^2 + cx + d$, & Parabolam deſignat cujus crura divergunt ab invicem & in contrarias partes infinite progrediuntur. Abſciſſa AB eſt ejus diameter & Species ejus ſunt quinque ſequentes.

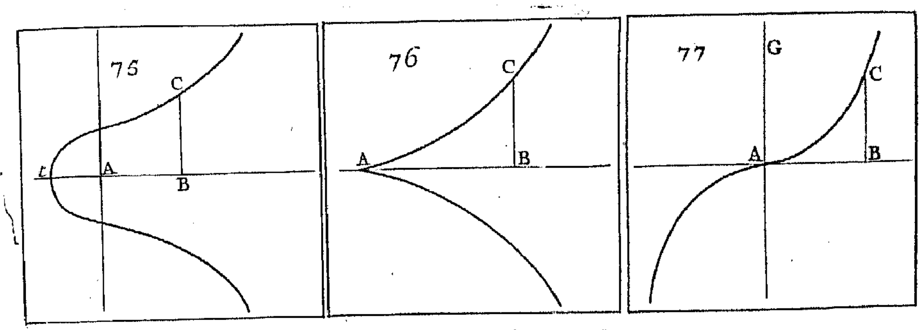
Si

Si æquationis $ax^3+bx^2+cx+d=0$, radices omnes $A\tau$, AT, At sunt reales & inæquales, figura est Parabola divergens Campaniformis cum *Ovali* ad verticem (*Fig*. 70, 71.) Et *Species* est *sexagesima septima*.

Si radices duæ sunt æquales, Parabola prodit vel *Nodata* contingendo Ovalem (*Fig*. 73,) vel *Punctata* ob Ovalem infinite parvam (*Fig*. 74.) Quæ duæ *Species* sunt *sexagesima octava* & *sexagesima nona*.

Si tres radices sunt æquales Parabola erit *Cuspidata* in vertice (*Fig*. 76.) Et hæc est Parabola *Neiliana* quæ vulgo *Semicubica* dicitur. Et est *Species septuagesima*.

Si radices duæ sunt impossibiles, habetur Parabola *Pura* campaniformis, (*Fig*. 74, 75,) *Speciem septuagesimam primam* constituens.



14. *De Parabola Cubica.*

In Quarto casu æquatio erat $y=ax^3+bx^2+cx+d$, & hæc æquatio Parabolam designat quæ crura habet contraria & *Cubica* dici solet (*Fig*.77.) Et sic *Species* omnino sunt *septuaginta duæ*.

V. *Genesis Curvarum per Umbras.*

Si in planum infinitum a puncto lucido illuminatum umbræ figurarum projiciantur, umbræ Sectionum Conicarum semper erunt Sectiones Conicæ, eæ Curvarum secundi Generis semper erunt Curvæ secundi Generis, eæ Curvarum tertii Generis semper erunt Curvæ tertii Generis, & sic deinceps in infinitum.

Et quemadmodum Circulus umbram projiciendo generat Sectiones omnes Conicas, sic Parabolæ quinque divergentes umbris suis generant & exhibent alias omnes secundi Generis Curvas; & sic Curvæ quædam simpliciores aliorum Generum inveniri possunt quæ alias omnes eorundem Generum Curvas umbris suis a puncto lucido in planum projectis formabunt.

De

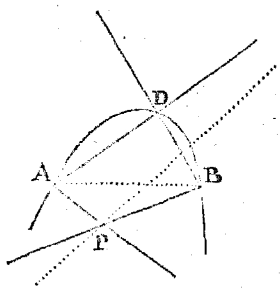
De Curvarum Punctis duplicibus.

Diximus Curvas ſecundi Generis a Linea recta in punctis tribus ſecari poſſe. Horum duo nonnunquam coincidunt. Ut cum Recta per Ovalem infinite parvam tranſit vel per concurſum duarum partium Curvæ ſe mutuo ſecantium vel in cuſpidem coeuntium ducitur. Et ſiquando Rectæ omnes in plagam cruris alicujus infiniti tendentes Curvam in unico tantum puncto ſecant, (ut fit in ordinatis Parabolæ *Carteſianæ* & Parabolæ cubicæ, nec non in rectis Abſciſſæ Hyperboliſmorum Hyperbolæ & Parabolæ parallelis) concipiendum eſt quod Rectæ illæ per alia duo Curvæ puncta ad infinitam diſtantiam ſita (ut ita dicam) tranſeunt. Hujuſmodi interſectiones duas coincidentes ſive ad finitam ſint diſtantiam ſive ad infinitam, vocabimus *Punctum Duplex*. Curvæ autem quæ habent Punctum Duplex deſcribi poſſunt per ſequentia Theoremata.

VI. *De Curvarum deſcriptione Organica.*

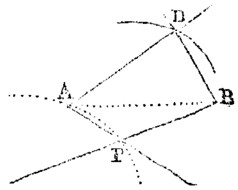
THEOR. I.

Si Anguli duo magnitudine dati PAD, PBD circa polos poſitione datos A, B rotentur, & eorum crura AP, BP concurſu ſuo P percurrant Lineam rectam; crura duo reliqua AD, BD concurſu ſuo D deſcribent Sectionem Conicam per polos A, B tranſeuntem: præterquam ubi Linea illa recta tranſit per polorum alterutrum A vel B, vel anguli BAD, ABD ſimul evaneſcunt, quibus in caſibus punctum D deſcribet Lineam rectam.



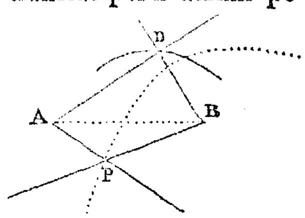
THEOR. II.

Si crura prima AP, BP concurſu ſuo P percurrant Sectionem Conicam per polum alterutrum A tranſeuntem, crura duo reliqua AD, BD concurſu su suo D describent Curvam secundi Generis per polum alterum B transeuntem & Punctum duplex habentem in polo primo A, per quem sectio Conica transit: præterquam ubi anguli BAD, ABD simul evanescunt, quo casu punctum D describet aliam sectionem Conicam per polum A transeuntem.



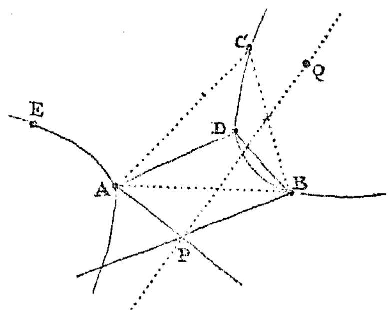
THEOR. III.

At si sectio Conica quam punctum P percurrit transeat per neutrum polorum A, B, punctum D describet Curvam secundi vel tertii generis Punctum duplex habentem. Et Punctum illud duplex in concursu crurum describentium, AD, BD invenietur ubi anguli BAP, ABP simul evanescunt. Curva autem descripta secundi erit Generis si anguli BAD, ABD simul evanescunt, alias erit tertii Generis & alia duo habebit Puncta duplicia in polis A & B.



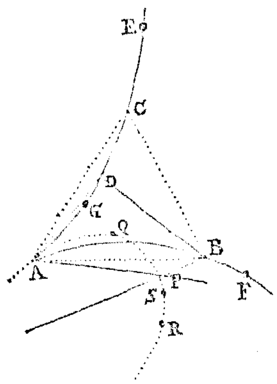
Sectionum Conicarum descriptio per data quinque puncta.

Jam sectio Conica determinatur ex datis ejus punctis quinque & per eadem sic describi potest. Dentur ejus puncta quinque A, B, C, D, E. Jungantur eorum tria quævis A, B, C, & trianguli ABC rotentur anguli duo quivis CAB, CBA circa vertices suos A & B, & ubi crurum AC, BC intersectio C successive applicatur ad puncta duo reliqua D, E, incidat intersectio crurum reliquorum AB & BA in puncta P & Q. Agatur & infinite producatur recta PQ, & anguli mobiles ita rotentur ut intersectio crurum AB, BA percurrat rectam PQ, & crurum reliquorum intersectio C describet propositam sectionem Conicam per *Theorema* primum.



Curvarum secundi generis Punctum duplex habentium descriptio per data septem puncta.

Curvæ omnes secundi generis Punctum Duplex habentes determinantur ex datis earum punctis septem quorum unum est Punctum illud duplex, & per eadem puncta sic describi possunt. Dentur Curvæ describendæ puncta quælibet septem A, B, C, D, E, F, G, quorum A est Punctum Duplex. Jungantur punctum A & alia duo quævis e punctis puta B & C; & trianguli ABC rotetur tum angulus CAB circa verticem suum A, tum angulorum reliquorum alteruter ABC circa verticem suum B. Et ubi crurum AC, BC concursus C successive applicatur ad puncta quatuor reliqua D, E, F, G incidat concursus crurum reliquorum AB & BA in puncta quatuor P, Q, R, S. Per puncta illa quatuor & quintum A describatur sectio Conica, & anguli præfati CAB, CBA ita rotentur ut crurum AB, BA concursus percurrat sectionem illam Conicam, & concursus reliquorum crurum AC, BC dedescribet Curvam propositam per *Theorema* secundum.



Si vice puncti C datur positione recta BC quæ Curvam describendam tangit in B, lineæ AD, AP coincident, & vice anguli DAP habebitur linea recta circa polum A rotanda.

Si Punctum duplex A infinite distat debebit Recta ad plagam puncti illius perpetuo dirigi & motu parallelo ferri interea dum angulus ABC circa polum B rotatur.

Describi etiam possunt hæ Curvæ paulo aliter per *Theorema* tertium, sed descriptionem simpliciorem posuisse sufficit.

Eadem methodo Curvas tertii, quarti & superiorum Generum describere licet, non omnes quidem sed quotquot ratione aliqua commoda per motum localem describi possunt. Nam Curvam aliquam secundi vel superioris generis Punctum duplex non habentem commode describere Problema est inter difficiliora numerandum.

VII. *Con-*

VII. *Conſtructio æquationum per deſcriptionem Curvarum.*

Curvarum uſus in Geometria eſt ut per earum interſectiones Problemata ſolvantur. Proponatur æquatio conſtruenda dimenſionum novem

$$x^9 * + bx^7 + cx^6 + dx^5 + ex^4 + \begin{array}{c} fx^3 \\ +m \end{array} + gx^2 + hx + k = 0.$$

Ubi b, c, d, &c. ſignificant quantitates quaſvis datas ſignis ſuis $+$ & $-$ affectas. Aſſumatur æquatio ad Parabolam cubicam $x^3 = y$, & æquatio prior, ſcribendo y pro x^3, evadet

$$y^3 + bxy^2 + cy^2 + dx^2y + exy + my + fx^3 + gx^2 + hx + k = 0,$$

æquatio ad Curvam aliam ſecundi Generis. Ubi m vel f deeſſe poteſt vel pro lubitu aſſumi. Et per harum Curvarum deſcriptiones & interſectiones dabuntur radices æquationis conſtruendæ. Parabolam cubicam ſemel deſcribere ſufficit.

Si æquatio conſtruenda per defectum duorum terminorum ultimorum hx & k reducatur ad ſeptem dimenſiones, Curva altera delendo m, habebit Punctum Duplex in principio Abſciſſæ, & inde facile deſcribi poteſt ut ſupra.

Si æquatio conſtruenda per defectum terminorum trium ultimorum $gx^2 + hx + k$ reducatur ad ſex dimenſiones, Curva altera delendo f evadet ſectio Conica.

Et ſi per defectum ſex ultimorum terminorum æquatio conſtruenda reducatur ad tres dimenſiones, incidetur in conſtructionem *Walliſianiam* per Parabolam cubicam & Lineam rectam.

Conſtrui etiam poſſunt æquationes per Hyperboliſmum Parabolæ cum diametro. Ut ſi conſtruenda ſit hæc æquatio dimenſionum novem termino penultimo carens,

$$a + cx^2 + dx^3 + ex^4 + fx^5 + \begin{array}{c} gx^6 \\ +m \end{array} + hx^7 + kx^8 + lx^9 = 0;$$

Aſſumatur æquatio ad Hyperboliſmum illum $x^2y = 1$, & ſcribendo y pro $\frac{1}{xx}$, æquatio conſtruenda vertetur in hanc

$$ay^3 + cy^2 + dxy^2 + ey + fxy + mx^2y + g + hx + kx^2 + lx^3 = 0,$$

quæ curvam ſecundi Generis deſignat cujus deſcriptione Problema ſolvetur. Et quantitatum m ac g alterutra hic deeſſe poteſt, vel pro lubitu aſſumi.

Per

Per Parabolam cubicam & Curvas tertii Generis conſtruuntur etiam æquationes omnes dimenſionum non pluſquam duodecim, & per eandem Parabolam & Curvas quarti Generis conſtruuntur omnes dimenſionum non pluſquam quindecim; Et ſic deinceps in infinitum. Et Curvæ illæ tertii, quarti & ſuperiorum Generum deſcribi ſemper poſſunt inveniendo eorum puncta per Geometriam planam. Ut ſi conſtruenda ſit æquatio
$x^{12} * + ax^{10} + bx^9 + cx^8 + dx^7 + ex^6 + fx^5 + gx^4 + hx^3 + ix^2 + kx + l = 0$,
& deſcripta habeatur Parabola Cubica; ſit æquatio ad Parabolam illam Cubicam $x^3 = y$, & ſcribendo y pro x^3, æquatio conſtruenda vertetur in hanc

$$\begin{array}{lllll} y^4 + axy^3 & + cx^2y^2 & + fx^2y & + ix^2 & = 0, \\ \quad + b & + dx & + gx & + kx & \\ & + e & + h & + l & \end{array}$$

quæ eſt æquatio ad Curvam tertii Generis cujus deſcriptione Problema ſolvetur. Deſcribi autem poteſt hæc Curva inveniendo ejus puncta per Geometriam planam, propterea quod indeterminata quantitas x non niſi ad duas dimenſiones aſcendit.

METHODUS DIFFERENTIALIS.

PROP. I.

SI figuræ curvilineæ Abſciſſa componatur ex quantitate quavis data A, *&* *quantitate indeterminata* x, *& Ordinata conſtet ex datis quotcunque quantitatibus* b, c, d, e, *&c. in totidem terminos hujus progreſſionis Geometricæ* x, x^2, x^3, x^4, *&c. reſpective ductis, & ad Abſciſſæ puncta totidem data erigantur Ordinatim applicatæ : dico quod Ordinatarum differentiæ primæ dividi poſſint per earum intervalla, & differentiarum ſic divi-*

ſarum differentiæ dividi poſſint per Ordinatarum binarum intervalla, & harum differentiarum ſic diviſarum differentiæ dividi poſſint per Ordinatarum ternarum intervalla, & ſic deinceps in infinitum.

Etenim ſi pro Abſciſſæ parte indeterminata x ponantur quantitates quævis datæ p, q, r, s, t, &c. ſucceſſive, & ad Abſciſſarum ſic datarum terminos erigantur Ordinatæ α, β, γ, δ, ε, &c. Hæ Abſciſſæ & Ordinatæ & Ordinatarum differentiæ diviſæ per Abſciſſarum differentias (quæ utique ſunt Ordinatarum intervalla) & quotorum differentiæ diviſæ per Ordinatarum alternarum differentias, & ſic deinceps, exhibentur per Tabulam ſequentem.

Abſciſſæ	Ordinatæ
$A+p$	$A+bp+cp^2+dp^3+ep^4=\alpha$
$A+q$	$A+bq+cq^2+dq^3+eq^4=\beta$
$A+r$	$A+br+cr^2+dr^3+er^4=\gamma$
$A+s$	$A+bs+cs^2+ds^3+es^4=\delta$
$A+t$	$A+bt+ct^2+dt^3+et^4=\varepsilon$
Diviſor. Diff. Ord.	Quoti per diviſionem prodeuntes.
$p-q)\ \alpha-\beta$	$b+c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+p^2q+pq^2+q^3}=\zeta$
$q-r)\ \beta-\gamma$	$b+c\times\overline{q+r}+d\times\overline{qq+qr+rr}+e\times\overline{q^3+q^2r+qr^2+r^3}=\eta$
$r-s)\ \gamma-\delta$	$b+c\times\overline{r+s}+d\times\overline{rr+rs+ss}+e\times\overline{r^3+r^2s+rs^2+s^3}=\theta$
$s-t)\ \delta-\varepsilon$	$b+c\times\overline{s+t}+d\times\overline{ss+st+tt}+e\times\overline{s^3+s^2t+st^2+t^3}=\kappa$
$p-r)\ \zeta-\eta$	$c+d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}=\lambda$
$q-s)\ \eta-\theta$	$c+d\times\overline{q+r+s}+e\times\overline{qq+qr+rr+qs+rs+ss}=\mu$
$r-t)\ \theta-\kappa$	$c+d\times\overline{r+s+t}+e\times\overline{rr+rs+ss+rt+st+tt}=\nu$
$p-s)\ \lambda-\mu$	$d+e\times\overline{p+q+r+s}=\xi.$
$q-t)\ \mu-\nu$	$d+e\times\overline{q+r+s+t}=\pi.$
$p-t)\ \xi-\pi$	$e=\sigma.$

PROP.

PROP. II.

Iiſdem poſitis, & quod numerus terminorum b, c, d, e, *&c. ſit finitus, dico quod Quotorum ultimus æqualis erit ultimo terminorum* b, c, d, e, *&c. et quod per Quotos reliquos dabuntur termini reliqui* b, c, d, e, *&c. et his datis dabitur Linea Curva generis Parabolici quæ per Ordinatarum omnium terminos tranſibit.*

Etenim in Tabula ſuperiore Quotus ultimus σ æqualis erat termino ultimo e. Et hic terminus ductus in ſummam datam $\overline{p+q+r+s}$, & ablatus de Quoto ξ relinquit terminum penultimum d. Et quantitates jam datæ $d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}$, ſi auferantur de Quoto λ, relinquunt terminorum antepenultimum c. Et quantitates jam datæ $c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+ppq+pqq+q^3}$, ſi auferantur de Quoto ζ, relinquunt terminum b. Et ſimili computo ſi plures eſſent termini, colligerentur omnes per Quotorum Ordines totidem. Deinde quantitates datæ $bp+cpp+dp^3+ep^4$, ſi ſubducantur de Ordinata prima α, relinquunt Abſciſſæ terminum primum A. Et quantitas $A+bx+cx^2+dx^3+ex^4+$ &c. eſt Ordinata Curvæ generis Parabolici quæ per Ordinatarum omnium datarum terminos tranſibit, exiſtente Abſciſſa $A+x$.

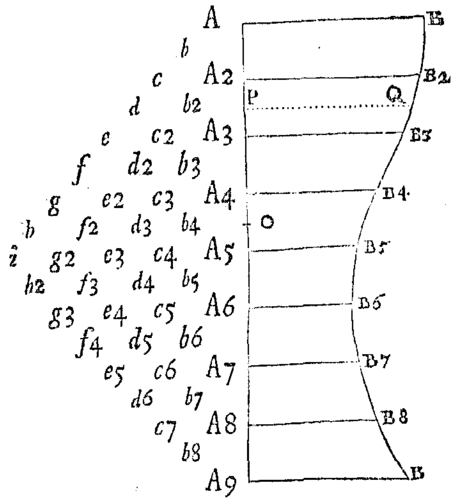
Ex his Propoſitionibus quæ ſequuntur facile colligi poſſunt.

PROP. III.

Si Recta aliqua AA9 *in æquales quotcunque partes* AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta diviſionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. tranſibit.*

Erectarum AB, A2B2, A3B3, &c. quære differentias Primas, b, $b2$, $b3$, &c. Secundas c, $c2$, $c3$, &c. Tertias d, $d2$, $d3$, &c. et ſic deinceps uſque dum veneris ad ultimam differentiam, quæ hic ſit i.

Tunc



Tunc incipiendo ab ultima differentia excerpe medias differentias in alternis Columnis vel Ordinibus differentiarum, & Arithmetica media inter duas medias reliquarum, Ordine pergendo usque ad Seriem primorum terminorum AB, A2B2, A3B3, &c. sint hæc $k, l, m, n, o, p, q, r, s$, &c. quorum ultimus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas differentias; antepenultimus mediam trium antepenultimarum differentiarum, & sic deinceps usque ad primum quod erit vel medius terminorum A, A2, A3, &c. vel Arithmeticus medius inter duos medios. Prius accidit ubi numerus terminorum A, A2, A3, &c. est impar; posterius ubi par.

CAS. I.

In Casu priori, sit A5B5 iste medius terminus, hoc est, $A5B5=k$, $\frac{b4+b5}{2}=l$, $c4=m$, $\frac{d3+d4}{2}=n$, $e3=o$, $\frac{f2+f3}{2}=p$, $g2=q$, $\frac{h+h2}{2}=r$, $i=s$. Et erecta Ordinatim applicata PQ, dic $A5P=x$; & duc terminos hujus Progressionis

$$1\times\frac{x}{1}\times\frac{x}{2}\times\frac{x^2-1}{3x}\times\frac{x}{4}\times\frac{x^2-4}{5x}\times\frac{x}{6}\times\frac{x^2-9}{7x}\times\frac{x}{8}\times\frac{x^2-16}{9x}\times\frac{x}{10}\times\frac{x^2-25}{11x}\times\frac{x}{12}\times\frac{x^2-36}{13x}\ \&c.$$

in se continuo; & orientur termini

$$1.\ x.\ \frac{x^2}{2}.\ \frac{x^3-x}{6}.\ \frac{x^4-x^2}{24}.\ \frac{x^5-5x^3+4x}{120}.\ \frac{x^6-5x^4+4x^2}{720}.\ \frac{x^7-14x^5+49x^3-36x}{5040}.\ \&c.$$

per quos si termini seriei k, l, m, n, o, p, &c. respective multiplicentur, aggregatum factorum $k+xl+\frac{x^2}{2}m+\frac{x^3-x}{6}n+\frac{x^4-x^2}{24}o+\frac{x^5-5x^3+4x}{120}p+$&c. erit longitudo Ordinatim applicatæ PQ.

CAS. II.

In Casu posteriori, sint A4B4, A5B5 duo medii termini, hoc est, sit $\frac{A4B4+A5B5}{2}=k$, $b4=l$, $\frac{c3+c4}{2}=m$, $d3=n$, $e2+e3=o$, $f2=p$, $\frac{g+g2}{2}=q$, & $h=r$.

& $b = r$. Et erecta Ordinatim applicata PQ, biseca A4A5 in O, & dicto OP $= x$, duc Terminos hujus Progressionis

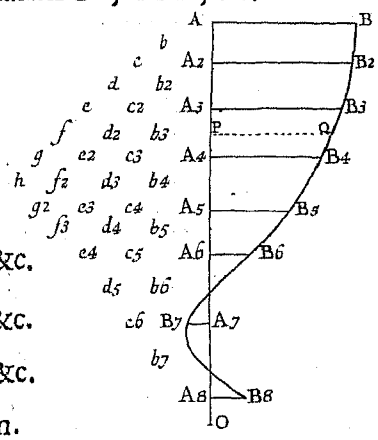
$1 \times \frac{x}{1} \times \frac{xx - \frac{1}{4}}{2x} \times \frac{x}{3} \times \frac{xx - \frac{9}{4}}{4x} \times \frac{x}{5} \times \frac{xx - \frac{25}{4}}{6x} \times \frac{x}{7} \times \frac{xx - \frac{49}{4}}{8x}$, &c. in se continuo; et orientur termini $1.\ x.\ \frac{4xx-1}{8}.\ \frac{4x^3-x}{24}.\ \frac{16x^4-40x^2+9}{384}.$ &c. per quos si termini series k, l, m, n, o, p, q, &c. respective multiplicentur, aggregatum factorum $k + xl + \frac{4x^2-1}{8}m + \frac{4x^3-x}{24}n + \frac{16x^4-40x^2+9}{384}o +$ &c. erit Longitudo Ordinatim applicatæ PQ.

Sed hic notandum est quod intervalla AA2, A2A3, A3A4, &c. hic supponantur esse unitates, & quod differentiæ colligi debent auferendo inferiores quantitates de superioribus, A2B2 de AB, A3B3 de A2B2, $b2$ de b, &c. et faciendo ut sint AB $-$ A2B2 $= b$, A2B2 $-$ A3B3 $= b2$, $b - b2 = c$, &c. adeoque quando differentiæ illæ hoc modo prodeunt negativæ signa earum mutanda sunt.

PROP. IV.

Si recta aliqua in partes quotcunque inæquales AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta divisionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. transibit.*

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Abscissam quamvis AA7 demitte Ordinatas perpendiculariter BA, B2A2, &c.



Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$,

$\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$,

$\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$,

$\frac{-A7B7 - A8B8}{A7A8} = b7$.

Deinde $\frac{b - b2}{AA3} = c$, $\frac{b2 - b3}{A2A4} = c2$, $\frac{b3 - b4}{A3A5} = c3$, &c.

Tunc $\frac{c - c2}{AA4} = d$, $\frac{c2 - c3}{A2A5} = d2$, $\frac{c3 - c4}{A3A6} = d3$, &c.

Et $\frac{d - d2}{AA5} = e$, $\frac{d2 - d3}{A2A6} = e2$, $\frac{d3 - d4}{A3A7} = e3$, &c.

Sic pergendum est ad ultimam differentiam.

Differentiis sit collectis & divisis per intervalla Ordinatim applicatarum; in alternis earum Columnis sive Seriebus vel Ordinibus excerpe medias, incipiendo ab ultima, & in reliquis Columnis excerpe media Arithmetica inter duas medias, pergendo usque ad seriem primorum terminorum, AB, A2B2, &c. Sunto hæc k, l, m, n, o, p, q, r, &c. quorum ultimus terminus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas; antepenultimus mediam trium antepenultimarum, &c. Et primus k erit media Ordinatim applicata, si numerus datorum punctorum est impar; vel medium Arithmeticum inter duas medias, si numerus eorum est par.

C A S. I.

In Casu priori, sit A4B4 ista media Ordinatim applicata, hoc est, sit $A4B4 = k$, $\frac{b3+b4}{2} = l$, $c3 = m$, $\frac{d2+d3}{2} = n$, $e2 = o$, $\frac{f+f2}{2} = p$, $g = q$. Et erecta Ordinatim applicata PQ, & in Basi AA5 sumpto quovis puncto O, dic $OP = x$, & duc in se gradatim terminos hujus Progressionis

$$1 \times \overline{x - OA4} \times \overline{x - \frac{OA3+OA5}{2}} \times \frac{\overline{x-OA3} \times \overline{x-OA5}}{\overline{x - \frac{1}{2}OA3 + OA5}} \times \overline{x - \frac{OA2+OA6}{2}} \times \text{\&c.}$$

et ortam Progressionem asserva; vel quod perinde est duc terminos hujus Progressionis

$$1 \times \overline{x - OA4} \times \overline{\overline{x-OA3} \times \overline{x-OA5}} \times \overline{\overline{x-OA2} \times \overline{x-OA6}} \times \overline{\overline{x-OA} \times \overline{x-OA7}} \times \text{\&c.}$$

in se gradatim, & terminos exinde ortos duc respective in terminos hujus Progressionis

$$1.\ x - \frac{+OA3+OA5}{2}.\ x - \frac{+OA2+OA6}{2}.\ x - \frac{+OA+OA7}{2}, \text{ \&c.}$$ et orientur termini intermedii tota Progressione existente

$$1.\ x - OA4.\ x^2 - \frac{+OA3+2OA4+OA5}{2}x + \frac{OA3+OA5}{2} \times OA4, \text{ \&c.}$$

Vel dic $OA = \alpha$, $OA2 = \beta$, $OA3 = \gamma$, $OA4 = \delta$, $OA5 = \epsilon$, $OA6 = \zeta$, $OA7 = \eta$: $\frac{OA3+OA5}{2} = \theta$, $\frac{OA2+OA6}{2} = \chi$, $\frac{OA+OA7}{2} = \lambda$. Et ex Progressione

$$1 \times \overline{x-\delta} \times \overline{\overline{x-\gamma} \times \overline{x-\epsilon}} \times \overline{\overline{x-\beta} \times \overline{x-\zeta}} \times \overline{\overline{x-\alpha} \times \overline{x-\eta}} \text{ \&c.}$$ collige terminos quibus multiplicatis per $1.\ x-\theta,\ x-\chi,\ x-\lambda$, &c collige alios terminos intermedios, tota serie prodeunte

$$1,\ x-\delta,\ x^2 - \overline{\delta+\theta}\,x + \delta\theta,\ x^3 - \overline{\delta+2\theta}\,x^2 + \overline{\gamma\epsilon+2\delta\theta}\,x - \gamma\delta\epsilon, \text{ \&c.}$$

per cujus terminos multiplica series k, l, m, n, o, &c. Et aggregatum productorum $k + \overline{x-\delta} \times l + \overline{x^2 - \overline{\delta+\theta}\,x + \delta\theta} \times m +$ &c. erit longitudo Ordinatim applicatæ PQ.

C A S.

C A S. II.

In Casu posteriori, sint $A4B4$, $A5B5$ duæ mediæ Ordinatim applicatæ, hoc est, $\frac{A4B4+A5B5}{2} = k$, $b4 = l$, $\frac{c3+c4}{2} = m$, $d3 = n$, $\frac{e2+e3}{2} = o$, $f2 = p$, &c. Et alternorum k, m, o, q, &c. Coefficientes orientur ex multiplicatione terminorum hujus Progressionis in se

$$1 \times \overline{x - OA4} \times \overline{x - OA5} \times \overline{x - OA3} \times \overline{x - OA6} \times \overline{x - OA2} \times \overline{x - OA7} \times \overline{x - OA} \times \overline{x - OA8} \text{ \&c.}$$

Et reliquorum Coefficientes ex multiplicatione horum per terminos hujus Progressionis

$$x - \frac{+OA4+OA5}{2},\ x - \frac{+OA3+OA6}{2},\ x - \frac{+OA2+OA7}{2},\ x - \frac{+OA+OA8}{2},\ \text{\&c.}$$

Hoc est, erit $k + \overline{x - \frac{+OA4+OA5}{2}} \times l + \overline{x^2 - \overline{OA4+OA5}x + OA4 \times OA5} \times m$, &c. Ordinatim applicata PQ,

vel $PQ = k + \begin{matrix} x \\ -\frac{1}{2}OA4 \\ -\frac{1}{2}OA5 \end{matrix} \times l + \begin{matrix} x \\ -OA4 \end{matrix} \times \begin{matrix} x \\ -OA5 \end{matrix} \times m + \begin{matrix} x \\ -OA4 \end{matrix} \times \begin{matrix} x \\ -OA5 \end{matrix} \times \begin{matrix} x \\ -\frac{1}{2}OA3 \\ -\frac{1}{2}OA6 \end{matrix} \times n$ &c.

Sive dic $x - \frac{+OA4+OA5}{2} = \pi$, $\overline{x - OA4} \times \overline{x - OA5} = \varrho$,

$\varrho \times \overline{x - \frac{+OA3+OA6}{2}} = \sigma$, $\varrho \times \overline{x - OA3} \times \overline{x - OA6} = \tau$,

$\tau \times \overline{x - \frac{+OA2+OA7}{2}} = \upsilon$, $\tau \times \overline{x - OA2} \times \overline{x - OA7} = \varphi$,

$\varphi \times \overline{x - \frac{+OA+OA8}{2}} = \chi$, $\varphi \times \overline{x - OA} \times \overline{x - OA8} = \psi$,

Et erit $k + \pi l + \varrho m + \sigma n + \tau o + \upsilon p + \varphi q + \chi r + \psi s = PQ$.

P R O P. V.

Datis aliquot terminis seriei cujuscunque ad data intervalla dispositis, invenire terminum quemvis intermedium quamproxime.

Ad rectam positione datam erigantur termini dati in dato angulo, interpositis datis intervallis, & per eorum puncta extima, per Propositiones præcedentes, ducatur linea Curva generis Parabolici. Hæc enim continget terminos omnes intermedios per seriem totam.

PROP.

PROP. VI.

Figuram quamcunque Curvilineam quadrare quamproxime, cujus Ordinatæ aliquot inveniri poſſunt.

Per terminos Ordinatarum ducatur linea Curva generis Parabolici ope Propoſitionum præcedentium. Hæc enim figuram terminabit quæ ſemper quadrari poteſt, et cujus Area æquabitur Areæ figuræ propoſitæ quamproxime.

SCHOLIUM.

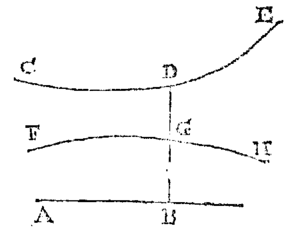
Utiles ſunt hæ Propoſitiones ad Tabulas conſtruendas per interpolationem Serierum, ut & ad ſolutiones Problematum quæ a quadraturis Curvarum dependent, præſertim ſi Ordinatarum intervalla & parva ſint & æqualia inter ſe, & Regulæ computentur, & in uſum reſerventur pro dato quocunque numero Ordinatarum. Ut ſi quatuor ſint Ordinatæ ad æqualia intervalla ſitæ, ſit A ſumma primæ & quartæ, B ſumma ſecundæ & tertiæ, & R intervallum inter primam & quartam, & Ordinata nova in medio omnium erit $\frac{9B-A}{16}$, & Area tota inter primam & quartam erit $\frac{A+3B}{8}R$.

Et nota quod ubi Ordinatæ ſtant ad æquales ab invicem diſtantias, ſumendo ſummas Ordinatarum quæ ab Ordinata media hinc inde æqualiter diſtant, & duplum Ordinatæ mediæ, componitur Curva nova cujus Area per pauciores Ordinatas determinatur, & æqualis eſt Areæ Curvæ prioris quam invenire oportuit. Quinetiam ſi pro Ordinatis novis ſumantur ſumma Ordinatæ primæ & ſecundæ, et ſumma tertiæ & quartæ, et ſumma quintæ & ſextæ, & ſic deinceps; vel ſi ſumantur ſumma trium primarum Ordinatarum, & ſumma trium proximarum, & ſumma trium quæ ſunt deinceps; vel ſi ſumantur ſummæ quaternarum Ordinatarum, vel ſummæ quinarum: Area Curvæ novæ æqualis erit Areæ Curvæ primo propoſitæ. Et ſic habitis Curvæ quadrandæ Ordinatis quotcunque quadratura ejus ad quadraturam Curvæ alterius per pauciores Ordinatas reducetur.

Per

Per data vero puncta quotcunque non ſolum Curvæ lineæ generis *Parabolici*, ſed etiam Curvæ aliæ innumeræ diverſorum generum duci poſſunt.

Sunto CDE, FGH Curvæ duæ Abſciſſam habentes communem AB, et Ordinatas in eadem recta jacentes BD, BG ; & relatio inter has Ordinatas definiatur per æquationem quamcunque. Dentur puncta quotcunque per quæ Curva CDE tranſire debet, & per æquationem illam dabuntur puncta totidem nova per quæ Curva FGH tranſibit. Per Propoſitiones ſuperiores deſcribatur Curva FGH generis *Parobolici* quæ per puncta illa omnia nova tranſeat, & per æquationem eandem dabitur Curva CDE quæ per puncta omnia primo data tranſibit.



F I N I S.